

VON NEUMANN CATEGORIES AND EXTENDED L^2 COHOMOLOGY

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ABSTRACT. In this paper we suggest a new general formalism for studying the L^2 invariants of polyhedra and manifolds. First, we examine generality in which one may apply the construction of *the extended abelian category*, which was suggested in [Fa, Fa1], using the ideas of P.Freyd [F]. This leads to the notions of a *finite von Neumann category* and of a *trace* on such category. Given a finite von Neumann category, we study *the extended L^2 homology and cohomology* theories with values in the abelian extension. Any trace on the initial category produces numerical invariants - the von Neumann dimension and the Novikov - Shubin numbers. Thus, we obtain *the local versions of the Novikov - Shubin invariants*, localized at different traces. In the "*abelian*" case this localization can be made more geometric: we show that any torsion object determines a "*divisor*" - a closed subspace of the space of the parameters. The divisors of torsion objects together with the information produced by the local Novikov - Shubin invariants may be used to study *multiplicities of intersections* of algebraic and analytic varieties (we discuss here only simple examples demonstrating this possibility). We compute explicitly the divisors and the von Neumann dimensions of the extended L^2 cohomology in the real analytic situation. We also give general formulae for the extended L^2 cohomology of a mapping torus. Finally, we show how one can define a De Rham version of the extended cohomology and prove a De Rham type theorem.

The classical approach to the L^2 cohomology theory, as developed in [A], [CG], [D], [G], consists in viewing it as a functor which assigns Hilbert modules over a von Neumann algebra to polyhedra and manifolds. This functor is also called *the reduced L^2 cohomology*, since in order to preserve the Hilbert module structure on the cohomology space one defines this cohomology as the quotient of the space of cocycles modulo the *closure* of the space of coboundaries.

This approach has certain well-known problems caused by the fact that Hilbert modules over a von Neumann algebra form only an *additive category*, and not an *abelian category*. The obtained cohomology theory has, in particular, difficulties in dealing with exact sequences. On the other hand, the reduced L^2 cohomology theory does not feel the phenomena "*zero in the continuous spectrum*", discovered by

S. Novikov and M. Shubin [NS], [NS1], which carries interesting topological information. This phenomenon is also related to the problem of absence of exactness; it was recently studied in [GS], [GS1] and in [LL].

In [Fa], [Fa1] there were constructed new homology and cohomology theories with values in *the extended abelian category*, containing (as the full subcategory of projectives) the usual category of finitely generated Hilbertian modules over a finite von Neumann algebra. These theories are called *extended L^2 homology and cohomology*. The *reduced L^2 (co)homology* appears as *the projective part* of the extended (co)homology. The extended homology and cohomology contain also *the torsion parts*; it was shown in [Fa, Fa1] that the torsion parts of the extended L^2 cohomology determine *the Novikov-Shubin invariants*. Moreover, it turned out (cf. [Fa, Fa1]) that the torsion parts of the extended cohomology produce also some new numerical invariants, which are independent of the Novikov-Shubin number. With the aid of these new invariants the Morse type inequalities of S. Novikov and M. Shubin [NS, NS1] were strengthened in [Fa, Fa1], so that the spectra near zero of the Laplacians also give quantitative information about the critical points.

The construction of the extended abelian category, which was suggested in [Fa, Fa1], used the general categorical study of P. Freyd [Fr]. It was described in [Fa, Fa1] in the simplest and smallest possible version, extending the additive category of finitely generated modules over a von Neumann algebra supplied with a fixed finite trace. In this small version the extension construction is equivalent to the purely algebraic alternative construction which was suggested later by W. Lück [L].

One of the main goals of the present paper is to investigate this extension construction in full generality. We try to find requirements on the category of Hilbert representations of a given $*$ -algebra under which the extension construction works. This leads to the notion of a *Hilbert category*, which is assumed to be closed under taking kernels and adjoint morphisms. We show that any Hilbert category can be canonically embedded in an abelian category. We establish a similar result for the Hilbertian representations as well, cf. §5. However, it is clear that it is impossible to apply this construction in the case of Frechet spaces or Banach spaces.

The further development of the theory can be made under additional assumption that the initial category is a *finite von Neumann category*. This guarantees that the torsion subcategory is also an abelian category. We show that any von Neumann algebra determines a von Neumann category; we discuss many interesting examples of von Neumann categories.

I should mention that (as I learned when the present paper was completely finished), P. Ghez, R. Lima, and J.E. Roberts in 1985 studied the notion of W^* -category (cf. [GLR]), which is equivalent to the notion of von Neumann category of this paper. In [GLR] they proved that many important results of the theory of von Neumann algebras (including, for example, the modular theory) can be generalized to the setting of von Neumann categories. Their motivation in [GLR] was completely different from ours, and this fact clearly explains why the intersection between the present paper and [GLR] is so little.

Next notion which appears to be extremely important for the formalism developed here is the notion of a *trace on a von Neumann category*. By the definition, a trace on a von Neumann category, is a compatible family of traces (in the usual sense of von Neumann algebras) on the rings of endomorphisms of all the objects. For

example, the category of finite dimensional vector spaces has only one trace up to normalization. Traces allow us to *localize* the notions of *von Neumann dimension* and the *Novikov - Shubin invariants*. The idea is to fix a category and to vary the trace on it. Given a trace on a finite von Neumann category, it determines the von Neumann dimension function (measuring sizes of the projective objects) and also the *Novikov - Shubin number* (measuring sizes of the torsion objects). We establish here the main properties of these function, which are mainly known, although in a different contexts.

As a special interesting example we consider here (in §4) the *abelian case*, which is given by the von Neumann category generated by fields of Hilbert spaces over a fixed compact space Z . This category allows to study *families*. Any chain complex formed by a sequence of vector bundles over Z and bundle maps between them determines a complex in this von Neumann category by considering the spaces of L^2 sections. We show that one may gain some interesting geometric information by studying the extended cohomology of the obtained complexes of L^2 -sections. The projective part of the extended cohomology (i.e. the reduced L^2 cohomology) can be easily explicitly computed. The torsion part of the extended cohomology contains much more information. In particular, it determines a *divisor* - a closed subset of Z . We prove that in the real analytic situation the divisor coincides with the subset of point $\xi \in Z$ where the fiberwise cohomology is not generic.

Simple examples computed in §4 show that one may hope to be able to study the *multiplicities of intersections* of algebraic and analytic varieties using the technique of extended cohomology.

The functors of extended homology and cohomology on the category of finite polyhedra are defined in section §6. As a coefficient systems for these theories we use arbitrary modules over the group ring in the extended abelian category (allowing torsion objects). This construction generalizes the well established practice of considering the regular representation only. We point out some basic properties of this construction, and then as an application, we compute explicitly the extended cohomology of mapping tori.

In the last section §7 we prove a De Rham type theorem for the extended cohomology. There are two candidates for the De Rham complex - one which is built on infinitely smooth forms and the other, which uses Sobolev spaces. We first show that the smooth De Rham complex is homotopy equivalent to the combinatorially defined Čech complex. However, one cannot use the smooth De Rham complex to define extended cohomology, since it is not clear how to construct an abelian category containing this complex. On the contrary, the Sobolev - De Rham complex (we show that it is homotopy equivalent to the smooth De Rham complex) belongs to the category of Hilbertian representations, and so the abelian extension construction of §5 applies to it.

I should add that M. Shubin was the first who initiated discussions of De Rham type theorems for the extended cohomology. His recent preprint [S] contains a De Rham type theorem different from the theorems of §7 of the present paper. I am very thankful to M. Shubin who sent to me a very preliminary version of [S].

I would like to mention also the preprint [CCMP], which discusses problems of similar nature, concerning relations between the spectral properties of smooth Laplacian and its combinatorial analogues corresponding to different triangulations.

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§1. Abelian extensions of Hilbert categories

In this section we will describe a construction of an abelian category extending in a canonical way a given category of $*$ -representations of an algebra with involution. It is based on the general categorical study of P.Freyd [F]. This construction appears to be of fundamental importance for the homology theory of topological algebras and their Hilbert representations. A special case of this construction was used in [Fa, Fa1] to introduce the notion of extended L^2 cohomology and to explain the homological nature of the Novikov-Shubin invariants.

1.1. Let \mathcal{A} be an algebra over \mathbb{C} having an involution which will be denoted by the star $*$. Recall, that this means that $*$: $\mathcal{A} \rightarrow \mathcal{A}$ is an involutive skew-linear map such that $(ab)^* = b^*a^*$ for $a, b \in \mathcal{A}$. A *Hilbert representation of \mathcal{A} (or a Hilbert module)* is a Hilbert space \mathcal{H} supplied with a left action of \mathcal{A} on \mathcal{H} by bounded linear maps such that for any $a \in \mathcal{A}$ holds

$$\langle ax, y \rangle = \langle x, a^*y \rangle \quad (1-1)$$

for all $x, y \in \mathcal{H}$. A *morphism* between Hilbert representations $\phi : \mathcal{H}_1 \rightarrow \mathcal{H}_2$ is a bounded linear map commuting with the action of the algebra \mathcal{A} .

The category of Hilbert representations is an additive category. It is easy to see that the kernel of any morphism $\phi : \mathcal{H}_1 \rightarrow \mathcal{H}_2$ between Hilbert representations in the sense of category theory coincides with $\phi^{-1}(0)$. However one observes that the categorical image of any morphism $\phi : \mathcal{H}_1 \rightarrow \mathcal{H}_2$ coincides with the *closure* of the set-theoretic image $\text{cl}(\phi(\mathcal{H}_1))$.

The category of Hilbert representations has the following important property: if $\mathcal{H}_1 \subset \mathcal{H}_2$ is a *closed* \mathcal{A} -invariant submodule then the orthogonal complement of \mathcal{H}_1 in \mathcal{H}_2 (denoted by \mathcal{H}_1^\perp) is also a closed \mathcal{A} submodule and the orthogonal projection $\pi : \mathcal{H}_2 \rightarrow \mathcal{H}_1$ is an \mathcal{A} -homomorphism. Thus, every closed submodule has a complement. Equivalently, any surjective morphism $\phi : \mathcal{H}_1 \rightarrow \mathcal{H}_2$ between Hilbert representations splits.

Next we observe that *the category of Hilbert representations is not an abelian category*. Indeed, any morphism $\phi : \mathcal{H}_1 \rightarrow \mathcal{H}_2$ between Hilbert representations which is injective and has dense image is both monomorphic and epimorphic; it is not an isomorphism unless ϕ is onto.

1.2. Our aim in this section is to construct, following ideas of P.Freyd [F], a larger *abelian* category, containing the category of Hilbert representations as a full subcategory.

In applications, we will always have some additional structures on the algebra \mathcal{A} (for example, topology, trace, etc.) and we will restrict ourselves to special additive subcategories of the category of Hilbert \mathcal{A} -representations. Our aim is to describe the construction of the extended abelian category in the most general unified way, in order to include all applications.

With this goal in mind, we will assume that an additive subcategory $\mathcal{C}_{\mathcal{A}}$ of the category of Hilbert representations over \mathcal{A} is specified. We will suppose that this additive subcategory $\mathcal{C}_{\mathcal{A}}$ has the following properties:

- (i) *The kernel of any morphism $\phi : \mathcal{H}_1 \rightarrow \mathcal{H}_2$ in $\mathcal{C}_{\mathcal{A}}$ and the natural inclusion $\ker \phi \rightarrow \mathcal{H}_1$ belong to $\mathcal{C}_{\mathcal{A}}$.*
- (ii) *For any morphism $\phi : \mathcal{H}_1 \rightarrow \mathcal{H}_2$ of $\mathcal{C}_{\mathcal{A}}$ the adjoint operator $\phi^* : \mathcal{H}_2 \rightarrow \mathcal{H}_1$ is also a morphism of $\mathcal{C}_{\mathcal{A}}$.*

Observe, that from (i) and (ii) follow:

- (iii) *The closure of the image $\text{cl}(\text{im } \phi)$ of any morphism $\phi : \mathcal{H}_1 \rightarrow \mathcal{H}_2$ in $\mathcal{C}_{\mathcal{A}}$ and also the natural projection $\mathcal{H}_2 \rightarrow \mathcal{H}_2 / \text{cl}(\text{im } \phi)$ belong to $\mathcal{C}_{\mathcal{A}}$.*
- (iv) *Suppose that $\mathcal{H}_1 \subset \mathcal{H}_2$ is a closed \mathcal{A} -invariant submodule. If \mathcal{H}_1 , \mathcal{H}_2 and the inclusion $\mathcal{H}_1 \rightarrow \mathcal{H}_2$ belong to $\mathcal{C}_{\mathcal{A}}$ then the orthogonal complement \mathcal{H}_1^\perp and the inclusion $\mathcal{H}_1^\perp \rightarrow \mathcal{H}_2$ belong to $\mathcal{C}_{\mathcal{A}}$.*

Definition. *An additive category $\mathcal{C}_{\mathcal{A}}$ of $*$ -representations of a $*$ -algebra \mathcal{A} satisfying the conditions (i) and (ii) above will be called Hilbert category.*

We will postpone discussion of examples until section 2.6.

In applications to topology, we will consider the situation when \mathcal{A} is the group algebra of a discrete group $\mathbb{C}[\pi]$.

1.3. Abelian extension of a Hilbert category. Returning to the general situation, given a Hilbert category $\mathcal{C}_{\mathcal{A}}$, we are going to define a bigger category $\mathcal{E}(\mathcal{C}_{\mathcal{A}})$, containing $\mathcal{C}_{\mathcal{A}}$ as a full subcategory.

An object of the category $\mathcal{E}(\mathcal{C}_{\mathcal{A}})$ is defined as a morphism $(\alpha : A' \rightarrow A)$ in the category $\mathcal{C}_{\mathcal{A}}$. Given a pair of objects $\mathcal{X} = (\alpha : A' \rightarrow A)$ and $\mathcal{Y} = (\beta : B' \rightarrow B)$ of $\mathcal{E}(\mathcal{C}_{\mathcal{A}})$, a morphism $\mathcal{X} \rightarrow \mathcal{Y}$ in the category $\mathcal{E}(\mathcal{C}_{\mathcal{A}})$ is an equivalence class of morphisms $f : A \rightarrow B$ of category $\mathcal{C}_{\mathcal{A}}$ such that $f \circ \alpha = \beta \circ g$ for some morphism $g : A' \rightarrow B'$ in $\mathcal{C}_{\mathcal{A}}$. Two morphisms $f : A \rightarrow B$ and $f' : A \rightarrow B$ of $\mathcal{C}_{\mathcal{A}}$ represent identical morphisms $\mathcal{X} \rightarrow \mathcal{Y}$ of $\mathcal{E}(\mathcal{C}_{\mathcal{A}})$ iff $f - f' = \beta \circ F$ for some morphism $F : A \rightarrow B'$ of category $\mathcal{C}_{\mathcal{A}}$. This defines an equivalence relation. The morphism $\mathcal{X} \rightarrow \mathcal{Y}$, represented by $f : A \rightarrow B$, is denoted by

$$[f] : (\alpha : A' \rightarrow A) \rightarrow (\beta : B' \rightarrow B) \quad \text{or by} \quad [f] : \mathcal{X} \rightarrow \mathcal{Y}.$$

The composition of morphisms is defined as the composition of the corresponding morphisms f in the category $\mathcal{C}_{\mathcal{A}}$.

The category $\mathcal{E}(\mathcal{C}_{\mathcal{A}})$ will be called *the abelian extension of the category $\mathcal{C}_{\mathcal{A}}$ or the extended category*, for short. We will see in Proposition 1.7 below that it is indeed abelian.

1.4. Excision. The following construction (which we will call *excision*) describes the typical changes which sometimes may be performed on objects of the extended category $\mathcal{E}(\mathcal{C}_{\mathcal{A}})$ without changing their isomorphism class.

Suppose that $\mathcal{X} = (\alpha : A' \rightarrow A)$ is an object of $\mathcal{E}(\mathcal{C}_{\mathcal{A}})$ and $P \subset A'$ is a closed \mathcal{A} submodule which belongs to $\mathcal{C}_{\mathcal{A}}$ and such that its image $Q = \alpha(P) \subset A$ is also closed. Using the assumptions (i)-(ii) concerning the category $\mathcal{C}_{\mathcal{A}}$ we obtain the following object $\mathcal{Y} = (\beta : P^\perp \rightarrow Q^\perp)$ where β is the composition of $\alpha|_{P^\perp}$ and the orthogonal

projection onto Q^\perp . We leave it as an easy exercise to show that \mathcal{Y} is isomorphic to the original object \mathcal{X} inside $\mathcal{E}(\mathcal{C}_A)$.

Using the excision, we may represent any isomorphism type in $\mathcal{E}(\mathcal{C}_A)$ by an injective morphism $(\alpha : A' \rightarrow A)$ (since we may always make an excision with respect to the kernel).

1.5. Here is another simple observation. An object $(\alpha : A' \rightarrow A)$ of $\mathcal{E}(\mathcal{C}_A)$ represents the zero object in $\mathcal{E}(\mathcal{C}_A)$ (i.e. is isomorphic to $(0 \rightarrow 0)$) if and only if α is surjective. In fact, if α is surjective then we may make an excision with respect to whole A' which produces the zero object. On the other hand, if $(\alpha : A' \rightarrow A)$ is isomorphic to the zero object, then from the definitions we obtain that there exists $F : A \rightarrow A'$ with $\alpha \circ F = 0$; thus α is surjective.

In order to show that $\mathcal{E}(\mathcal{C}_A)$ is an abelian category we will need the following statement.

1.6. Proposition. Let $[f] : (\alpha : A' \rightarrow A) \rightarrow (\beta : B' \rightarrow B)$ be a morphism in $\mathcal{E}(\mathcal{C}_A)$. Then its kernel is represented by

$$[k] : (\gamma : P' \rightarrow P) \rightarrow (\alpha : A' \rightarrow A),$$

where

$$\begin{array}{ccc} P & \xrightarrow{k} & A \\ f' \downarrow & & \downarrow f \\ B' & \xrightarrow{\beta} & B \end{array} \quad \text{and} \quad \begin{array}{ccc} P' & \xrightarrow{k'} & A' \\ f'' \downarrow & & \downarrow f \circ \alpha \\ B' & \xrightarrow{\beta} & B \end{array} \quad (1-2)$$

are the pullbacks of the diagrams

$$\begin{array}{ccc} & A & \\ & \downarrow f & \\ B' & \xrightarrow{\beta} & B \end{array} \quad \text{and} \quad \begin{array}{ccc} & A' & \\ & \downarrow f \circ \alpha & \\ B' & \xrightarrow{\beta} & B \end{array} \quad (1-3)$$

correspondingly, and $\gamma : P' \rightarrow P$ is the canonical map, induced by the obvious map of the right diagram (1-2) into the left one.

The cokernel of the above morphism $[f]$ is represented by

$$[\text{id}_B] : (\beta : B' \rightarrow B) \rightarrow ((\beta, -f) : B' \oplus A \rightarrow B).$$

Proof. Note that pullbacks are defined as follows: P is the kernel of the morphism $f \oplus -\beta : A \oplus B' \rightarrow B$, and P' is defined similarly; they exist by our assumption (i) above.

To prove the statement concerning the kernels, note first that clearly $[f] \circ [k] = 0$. Suppose that $(\gamma : C' \rightarrow C)$ is an object of $\mathcal{E}(\mathcal{C}_A)$ and that we are given a morphism

$$[g] : (\gamma : C' \rightarrow C) \rightarrow (\alpha : A' \rightarrow A)$$

such that $[f] \circ [g] = 0$. We want to show that $[g]$ could be factorized uniquely through $[k]$. Since $f \circ g$ represents a zero morphism in $\mathcal{E}(\mathcal{C}_A)$, there exists a morphism

$F : C \rightarrow B'$ such that $\beta \circ F = f \circ g$. The pair of morphisms (g, F) determines a morphism $h : C \rightarrow P$ with $k \circ h = g$. From the definition of morphisms in $\mathcal{E}(\mathcal{C}_{\mathcal{A}})$ we know that there is a morphism $g' : C' \rightarrow A'$ with $\alpha \circ g' = g \circ \gamma$; thus, the pair of morphisms $(g', F \circ \gamma)$ determines a morphism $h' : C' \rightarrow P'$ showing that we have a morphism in $\mathcal{E}(\mathcal{C}_{\mathcal{A}})$

$$[h] : (\gamma : C' \rightarrow C) \rightarrow (k : P' \rightarrow P)$$

which gives the desired factorization $[g] = [k] \circ [h]$.

To show uniqueness of the above factorization, we may assume in the notations of the previous paragraph that $[k] \circ [g] = 0$ and then prove that then $[g] = 0$. In fact, $[k] \circ [g] = 0$ means (according to our definitions) that there exists a morphism $G : C \rightarrow A'$ with $k \circ G = \alpha \circ G$. The pair of morphisms $G : C \rightarrow A'$ and $C \xrightarrow{g} P \rightarrow B'$ determine a morphism $G' : C \rightarrow P'$ with $\gamma \circ G' = g$; thus $[g] = 0$.

The second statement concerning the cokernels can be checked similarly.

1.7. Proposition. $\mathcal{E}(\mathcal{C}_{\mathcal{A}})$ is an abelian category.

Proof. We already know that any morphism in $\mathcal{E}(\mathcal{C}_{\mathcal{A}})$ has a kernel and a cokernel. We have to prove that any morphism $[f] : (\alpha : A' \rightarrow A) \rightarrow (\beta : B' \rightarrow B)$ in $\mathcal{E}(\mathcal{C}_{\mathcal{A}})$ which is both monomorphic and epimorphic is an isomorphism in $\mathcal{E}(\mathcal{C}_{\mathcal{A}})$. We may assume without loss of generality, that both α and β are injective. We will use the notations introduced in Proposition 1.6. The morphism $\gamma : P' \rightarrow P$ is an isomorphism and we have a exact sequences

$$\begin{aligned} 0 \rightarrow P &\xrightarrow{(k, f')} A \oplus B' \xrightarrow{(f, -\beta)} B \rightarrow 0 \\ 0 \rightarrow P' &\xrightarrow{(k', f'')} A' \oplus B' \xrightarrow{(f', -\text{id})} B' \rightarrow 0 \end{aligned}$$

Thus we may represent the morphism $[f]$ as the composite of two excisions

$$(\alpha : A' \rightarrow A) \xrightarrow{\cong} \left(\begin{bmatrix} \alpha & 0 \\ 0 & 1 \end{bmatrix} : A' \oplus B' \rightarrow A \oplus B' \right) \xrightarrow[\cong]{(f, -\beta)} (\beta : B' \rightarrow B)$$

Here the excision on the right is performed with respect to the closed submodule $P' \subset A' \oplus B'$; it is mapped by $\begin{bmatrix} \alpha & 0 \\ 0 & 1 \end{bmatrix}$ onto the closed submodule $P \subset A \oplus B'$.

This completes the proof. \square

1.8. Embedding of $\mathcal{C}_{\mathcal{A}}$ into $\mathcal{E}(\mathcal{C}_{\mathcal{A}})$. Given an object $A \in \text{ob}(\mathcal{C}_{\mathcal{A}})$ one defines the following object $(0 \rightarrow A) \in \text{ob}(\mathcal{E}(\mathcal{C}_{\mathcal{A}}))$ of the extended category. Since any morphism $f : A \rightarrow B$ determines a morphism $[f] : (0 \rightarrow A) \rightarrow (0 \rightarrow B)$ in the extended category, we obtain an embedding $\mathcal{C}_{\mathcal{A}} \rightarrow \mathcal{E}(\mathcal{C}_{\mathcal{A}})$. In fact, it is easy to see that this embedding is full.

We want to characterize the objects of the extended category which are isomorphic in $\mathcal{E}(\mathcal{C}_{\mathcal{A}})$ to objects coming from $\mathcal{C}_{\mathcal{A}}$ in intrinsic terms:

1.9. Proposition. *An object $\mathcal{X} \in \text{ob}(\mathcal{E}(\mathcal{C}_{\mathcal{A}}))$ is projective if and only if it is isomorphic in $\mathcal{E}(\mathcal{C}_{\mathcal{A}})$ to an object of the form $(0 \rightarrow A)$, where $A \in \text{ob}(\mathcal{C}_{\mathcal{A}})$.*

Proof. Let $\mathcal{X} = (\beta : B' \rightarrow B)$; by 1.4 we may assume that β is injective. Using Proposition 1.6 we see that the sequence

$$0 \rightarrow (0 \rightarrow B') \xrightarrow{[\beta]} (0 \rightarrow B) \xrightarrow{[\text{id}]} \mathcal{X} \rightarrow 0$$

is exact and so it splits if \mathcal{X} is supposed to be projective. Any splitting is given by a homomorphism $r : B \rightarrow B$ such that $r \circ \beta = 0$ and $\text{id} - r = \beta \circ F$ for some $F : B \rightarrow B'$. It now follows that the image of β coincides with the kernel of r and so the image of β is closed. Thus, we may make excision with respect to the whole B' and hence by 1.4 we obtain that \mathcal{X} is isomorphic in $\mathcal{E}(\mathcal{C}_{\mathcal{A}})$ to $(0 \rightarrow A)$, where $A = B/\beta(B')$.

Conversely, let $\mathcal{X} = (0 \rightarrow A)$ and suppose that $[f] : (\alpha : A' \rightarrow A) \rightarrow (\beta : b' \rightarrow B)$ is an epimorphism in $\mathcal{E}(\mathcal{C}_{\mathcal{A}})$. We want to show that arbitrary morphism $[g] : (0 \rightarrow A) \rightarrow (\beta : B' \rightarrow B)$ can be lifted in $(\alpha : C' \rightarrow C)$. By Proposition 1.6, the morphism $(\beta, -f) : B' \oplus C \rightarrow B$ is an epimorphism in $\mathcal{C}_{\mathcal{A}}$ and so there exists a morphism $G : A \rightarrow B' \oplus C$ with $(-\beta, f) \circ G = g$. Writing $G = g_1 \oplus g_2$, where $g_1 : A \rightarrow B'$ and $g_2 : A \rightarrow C$, we obtain $[g] = [f] \circ [g_2]$ is $\mathcal{E}(\mathcal{C}_{\mathcal{A}})$. \square .

Using embedding $\mathcal{C}_{\mathcal{A}} \rightarrow \mathcal{E}(\mathcal{C}_{\mathcal{A}})$ and Proposition 1.9, any chain complex in $\mathcal{C}_{\mathcal{A}}$ can be viewed as a *projective chain complex in category* $\mathcal{E}(\mathcal{C}_{\mathcal{A}})$. Since $\mathcal{E}(\mathcal{C}_{\mathcal{A}})$ is abelian, we may define the homology of the chain complex as a graded object of $\mathcal{E}(\mathcal{C}_{\mathcal{A}})$. The corresponding homology will be called *the extended homology of the chain complex*. The following statement contains the explicit computation of the extended homology.

1.10. Proposition. *Given a chain complex C in a Hilbert category $\mathcal{C}_{\mathcal{A}}$*

$$\cdots \rightarrow C_{i+1} \xrightarrow{\partial} C_i \xrightarrow{\partial} C_{i-1} \xrightarrow{\partial} \cdots, \quad (1-4)$$

the homology of this complex, viewed as an object of the extended category $\mathcal{E}(\mathcal{C}_{\mathcal{A}})$, equals

$$H_i(C) = (\partial : C_{i+1} \rightarrow Z_i) \in \text{ob}(\mathcal{E}(\mathcal{C}_{\mathcal{A}})), \quad (1-5)$$

where Z_i is the space of the cycles, $Z_i = \ker[\partial : C_i \rightarrow C_{i-1}]$.

Proof. It is an easy exercise based on applying Proposition 1.6.

The term *extended homology* intends to emphasize distinction between the present approach and the well established practice of assigning the following (sometimes it is called *reduced*) homology to a chain complex of Hilbert spaces

$$Z_i / \text{cl}(\text{im}[\partial C_{i+1} \rightarrow C_i]).$$

The extended homology is *larger* since there is natural epimorphism in $\mathcal{E}(\mathcal{C}_{\mathcal{A}})$

$$H_i(C) = (\partial : C_{i+1} \rightarrow Z_i) \rightarrow Z_i / \text{cl}(\text{im}[\partial C_{i+1} \rightarrow C_i]).$$

1.11. Functor of projective part. Here we will introduce the notion which allows to express the relation between the reduced and extended homology in a precise way.

Let $\mathcal{X} = (\alpha : A' \rightarrow A)$ be an object of the extended category $\mathcal{E}(\mathcal{C}_{\mathcal{A}})$. Its *projective part* is defined as the following object of $\mathcal{C}_{\mathcal{A}}$

$$P(\mathcal{X}) = A / \text{cl}(\text{im}(\alpha)). \quad (1-6)$$

Clearly, any morphism $f : \mathcal{X} \rightarrow \mathcal{Y}$ in $\mathcal{E}(\mathcal{C}_{\mathcal{A}})$ induces a morphism in $\mathcal{C}_{\mathcal{A}}$ $P(f) : P(\mathcal{X}) \rightarrow P(\mathcal{Y})$ between the projective parts and so *we have well defined functor*

$$P : \mathcal{E}(\mathcal{C}_{\mathcal{A}}) \rightarrow \mathcal{C}_{\mathcal{A}}. \quad (1-7)$$

For a short exact sequence in $\mathcal{E}(\mathcal{C}_{\mathcal{A}})$

$$0 \rightarrow \mathcal{X}' \rightarrow \mathcal{X} \rightarrow \mathcal{X}'' \rightarrow 0,$$

the corresponding sequence of projective parts

$$0 \rightarrow P(\mathcal{X}') \rightarrow P(\mathcal{X}) \rightarrow P(\mathcal{X}'') \rightarrow 0 \quad (1-8)$$

may be *not exact* in the middle term (although it is exact in the other places). Thus the functor of projective part *is not half-exact*.

Using the functor of projective part we may express the rule which explains how the extended homology determines the reduced homology. Namely, *the reduced homology coincides with the projective part of the extended homology*.

Non-exactness of the functor P of projective part explains well-known difficulties with exact sequences of the reduced homology. The extended homology theory is free of these problems since it assumes its values in an honest abelian category.

1.12. Remark. Suppose that a Hilbert category $\mathcal{C}_{\mathcal{A}}$ is a subcategory of another Hilbert category $\mathcal{C}'_{\mathcal{A}}$. Then we have an obvious functor $F : \mathcal{E}(\mathcal{C}_{\mathcal{A}}) \rightarrow \mathcal{E}(\mathcal{C}'_{\mathcal{A}})$ between the extended categories. In general, it may happen that $F([f]) = 0$ for a nonzero morphism $[f]$ of $\mathcal{E}(\mathcal{C}_{\mathcal{A}})$.

However, *if a Hilbert category $\mathcal{C}_{\mathcal{A}}$ is a full subcategory of another Hilbert category $\mathcal{C}'_{\mathcal{A}}$ then the above functor F represents the extended abelian category $\mathcal{E}(\mathcal{C}_{\mathcal{A}})$ as a full subcategory of the extended category $\mathcal{E}(\mathcal{C}'_{\mathcal{A}})$.*

§2. Von Neumann categories

Here we will introduce a notion of von Neumann category which naturally generalizes the notion of von Neumann algebra. The notion of von Neumann category is equivalent (but it is not identical) to the notion of W^* -category which was first introduced in [GLR]. It seems possible to develop the theory of von Neumann categories to a large extent in a way similar to the classical theory of von Neumann algebras. We will however mention here only basic properties, which are used in this paper.

Our interest in von Neumann categories is based on the fact, that (as we will see later in section §3) the extended abelian category $\mathcal{E}(\mathcal{C}_{\mathcal{A}})$ has many important additional properties in case when the original category $\mathcal{C}_{\mathcal{A}}$ is a von Neumann category.

2.1. Von Neumann categories of Hilbert representations. We will say that a category of Hilbert representations $\mathcal{C}_{\mathcal{A}}$ of a $*$ -algebra \mathcal{A} is a *von Neumann category* if it satisfies besides conditions (i) and (ii) of section 1, the following condition:

- (v) *for any pair of representations $\mathcal{H}_1, \mathcal{H}_2 \in \text{ob}(\mathcal{C}_{\mathcal{A}})$, the corresponding set of morphisms $\text{Hom}_{\mathcal{C}_{\mathcal{A}}}(\mathcal{H}_1, \mathcal{H}_2)$ is a weakly closed subspace in the space of all bounded linear operators between \mathcal{H}_1 and \mathcal{H}_2 .*

Recall, that the weak topology on the space of bounded linear operators $f : \mathcal{H}_1 \rightarrow \mathcal{H}_2$ is given by the family of seminorms

$$p_{\phi, x}(f) = |\langle \phi, f(x) \rangle|, \quad \text{where } \phi \in \mathcal{H}_2, \quad x \in \mathcal{H}_1.$$

In particular, for any object $\mathcal{H} \in \text{ob}(\mathcal{C}_{\mathcal{A}})$ of a von Neumann category the set of endomorphisms $\text{Hom}_{\mathcal{C}_{\mathcal{A}}}(\mathcal{H}, \mathcal{H})$ (*i.e. the commutant*) is a von Neumann algebra.

Note also that the same notion of von Neumann category will be obtained if we require that the set of morphisms $\text{Hom}_{\mathcal{C}_A}(\mathcal{H}_1, \mathcal{H}_2)$ is strongly (instead if weakly) closed in the space of all bounded linear operators between \mathcal{H}_1 and \mathcal{H}_2 . This follows easily from von Neumann density theorem (cf. [Di], part I, chapter 3, §4) applied to the von Neumann algebra $\text{Hom}_{\mathcal{C}_A}(\mathcal{H}_1 \oplus \mathcal{H}_2, \mathcal{H}_1 \oplus \mathcal{H}_2)$.

Observe, that in a von Neumann category \mathcal{C}_A the notions of isomorphism and unitary equivalence (i.e. \mathcal{C}_A -isomorphism, preserving the scalar products) coincide. Namely, if $f : \mathcal{H}_1 \rightarrow \mathcal{H}_2$ is an \mathcal{C}_A -isomorphism then we can find $T \in \text{Hom}_{\mathcal{C}_A}(\mathcal{H}_1, \mathcal{H}_2)$ such that $T^2 = f^*f$, $T^* = T$, $T > 0$ cf. [Di], part I, chapter 1, §2. Then the map $g = fT^{-1} : \mathcal{H}_1 \rightarrow \mathcal{H}_2$ is a unitary equivalence.

2.2. Finite objects. We will say that an object $\mathcal{H} \in \text{ob}(\mathcal{C}_A)$ of a von Neumann category is *finite* if the von Neumann algebra of its endomorphisms $\text{Hom}_{\mathcal{C}_A}(\mathcal{H}, \mathcal{H})$ is finite. Recall that finiteness of the von Neumann algebra $\text{Hom}_{\mathcal{C}_A}(\mathcal{H}, \mathcal{H})$ is equivalent to the fact that *any closed \mathcal{C}_A -submodule $\mathcal{H}_1 \subset \mathcal{H}$ which is isomorphic to \mathcal{H} in \mathcal{C}_A , coincides with \mathcal{H} .* (Cf. [Di], part III, chapter 8, §1.)

If \mathcal{C}_A is a von Neumann category, and if $\mathcal{H} \in \text{ob}(\mathcal{C}_A)$ is finite then any submodule of \mathcal{H} in \mathcal{C}_A is also finite. The direct sum of finite objects is also finite. Thus, all finite objects of any von Neumann category form a von Neumann subcategory.

A von Neumann category will be called *finite* if all its objects are finite.

A von Neumann category will be called *semi-finite* if any its object \mathcal{H} can be represented as a union of an increasing sequence of closed finite subobjects $\mathcal{H}_n \subset \mathcal{H}$.

2.3. Lemma. *Let \mathcal{C}_A be a von Neumann category and let $\phi : \mathcal{H}_1 \rightarrow \mathcal{H}_2$ be an injective morphism of \mathcal{C}_A having dense image. Then the Hilbert representations \mathcal{H}_1 and \mathcal{H}_2 are isomorphic in \mathcal{C}_A .*

Proof. Consider $f^*f : \mathcal{H}_1 \rightarrow \mathcal{H}_1$; because of property (ii) of section (i) we know that f^*f belongs to $\text{Hom}_{\mathcal{C}_A}(H_1, H_1)$. Thus f^*f is a positive element of von Neumann algebra $\text{Hom}_{\mathcal{C}_A}(H_1, H_1)$, and so there exists $T \in \text{Hom}_{\mathcal{C}_A}(H_1, H_1)$ such that $T^* = T$, $T > 0$ and $T^2 = f^*f$ (cf. [Di]).

Consider the operator $fT^{-1} : TH_1 \rightarrow H_2$. It is densely defined and bounded. So it can be uniquely extended to a bounded operator $fT^{-1} : H_1 \rightarrow H_2$. Now, we claim that the constructed operator $fT^{-1} : H_1 \rightarrow H_2$ belongs to $\text{Hom}_{\mathcal{C}_A}(H_1, H_2)$. To show this, define

$$A_\epsilon = f \circ \int_\epsilon^\infty \mu_\epsilon(\lambda) dE_\lambda$$

where $T = \int_0^\infty \lambda dE_\lambda$ and the real valued continuous function $\mu_\epsilon(\lambda)$ is given by $\mu_\epsilon(\lambda) = \lambda^{-1}$ for $\lambda \geq \epsilon$ and $\mu_\epsilon(\lambda) = \epsilon^{-1}$ for $\lambda \leq \epsilon$. Then we have: A_ϵ belongs to $\text{Hom}_{\mathcal{C}_A}(H_1, H_2)$ and A_ϵ converges to $f \circ T^{-1}$ weakly as $\epsilon \rightarrow 0$. \square

2.4. Proposition. *Any finite object $\mathcal{H} \in \text{ob}(\mathcal{C}_A)$ of a von Neumann category \mathcal{C}_A has the following property. Suppose that $\phi : \mathcal{H}_1 \rightarrow \mathcal{H}$ is an injective morphism in \mathcal{C}_A such that its image $\text{im } \phi$ is dense in \mathcal{H} . Then for any nonzero Hilbert representation $\mathcal{H}'_1 \in \text{ob}(\mathcal{C}_A)$ and any injective morphism $\psi : \mathcal{H}'_1 \rightarrow \mathcal{H}$ in \mathcal{C}_A , the intersection $\text{im } \phi \cap \text{im } \psi$ is nonzero.*

The role of this property will become clear in the next section where we will study the torsion objects of the extended category.

Proof. Suppose that $\mathcal{C}_{\mathcal{A}}$ is a von Neumann category and $\mathcal{H}_2 \in \text{ob}(\mathcal{C}_{\mathcal{A}})$ is finite. Let $\phi : \mathcal{H}_1 \rightarrow \mathcal{H}_2$ be an injective morphism of $\mathcal{C}_{\mathcal{A}}$ with dense image and let $\psi : \mathcal{H}'_1 \rightarrow \mathcal{H}_2$ be another injective morphism of $\mathcal{C}_{\mathcal{A}}$ such that \mathcal{H}'_1 is nonzero and the intersection of the images of ϕ and ψ is 0.

First, we want to find a smaller nontrivial $\mathcal{C}_{\mathcal{A}}$ -submodule $\mathcal{H}''_1 \subset \mathcal{H}'_1$ such that the image $\psi(\mathcal{H}''_1)$ is closed. To do so, consider the spectral decomposition

$$\psi^* \psi = \int_0^\infty \lambda dE_\lambda$$

and define $\mathcal{H}''_1 = (E_\epsilon \mathcal{H}'_1)^\perp$ for some small $\epsilon > 0$. This ϵ must be chosen so small that \mathcal{H}''_1 is nonzero. Then for any $x \in \mathcal{H}''_1$ we have $|\psi(x)| \geq \sqrt{\epsilon} \cdot |x|$. Thus, the image of $\psi|_{\mathcal{H}''_1}$ is closed.

Let $\mathcal{H}'_2 \subset \mathcal{H}_2$ denote $\psi(\mathcal{H}''_1)^\perp$ and let $\pi : \mathcal{H}_2 \rightarrow \mathcal{H}'_2$ denote the orthogonal projection. The composite $\pi \circ \phi : \mathcal{H}_1 \rightarrow \mathcal{H}'_2$ is injective with dense image. Thus \mathcal{H}_1 and \mathcal{H}'_2 are isomorphic in $\mathcal{C}_{\mathcal{A}}$ by Proposition 2.4. Similarly, applying Proposition 2.4 again we see that \mathcal{H}_1 and \mathcal{H}_2 are isomorphic. Thus we obtain that there exists an isomorphism $\mathcal{H}_2 \rightarrow \mathcal{H}'_2$ in $\mathcal{C}_{\mathcal{A}}$ which contradicts finiteness of \mathcal{H}_2 . \square

2.5. Constructing von Neumann categories. We will describe here two completion constructions which allow to construct many interesting examples of von Neumann categories.

(a) *Completing the set of morphisms.* Suppose that $\mathcal{C}_{\mathcal{A}}$ is a category whose objects have the structure of Hilbert representations of a $*$ -algebra \mathcal{A} and such that the morphisms of $\mathcal{C}_{\mathcal{A}}$ are represented by bounded linear maps, i.e.

$$\text{Hom}_{\mathcal{C}_{\mathcal{A}}}(\mathcal{H}_1, \mathcal{H}_2) \subset \mathcal{L}(\mathcal{H}_1, \mathcal{H}_2).$$

We will assume that $\mathcal{C}_{\mathcal{A}}$ is *self-adjoint*, i.e. condition (ii) of section 1.2 is satisfied.

Consider category $\tilde{\mathcal{C}}_{\mathcal{A}}$ which has the same objects as $\mathcal{C}_{\mathcal{A}}$ and for any two objects $\mathcal{H}_1, \mathcal{H}_2 \in \text{ob}(\tilde{\mathcal{C}}_{\mathcal{A}})$ the set of morphisms $\text{Hom}_{\tilde{\mathcal{C}}_{\mathcal{A}}}(\mathcal{H}_1, \mathcal{H}_2)$ in $\tilde{\mathcal{C}}_{\mathcal{A}}$ is defined as the closure of $\text{Hom}_{\mathcal{C}_{\mathcal{A}}}(\mathcal{H}_1, \mathcal{H}_2)$ with respect to the weak operator topology.

In order to check that this defines a category we have to show that if $f : \mathcal{H}_1 \rightarrow \mathcal{H}_2$ is the limit (in the weak operator topology) of a net $f_\lambda \in \text{Hom}_{\mathcal{C}_{\mathcal{A}}}(\mathcal{H}_1, \mathcal{H}_2)$ and if $g : \mathcal{H}_2 \rightarrow \mathcal{H}_3$ is the limit of a net $g_\mu \in \text{Hom}_{\mathcal{C}_{\mathcal{A}}}(\mathcal{H}_2, \mathcal{H}_3)$ then the bounded linear map $g \circ f : \mathcal{H}_1 \rightarrow \mathcal{H}_3$ belongs to the weak closure of $\text{Hom}_{\mathcal{C}_{\mathcal{A}}}(\mathcal{H}_1, \mathcal{H}_3)$ in $\mathcal{L}(\mathcal{H}_1, \mathcal{H}_3)$, which we denote C .

We know that the composition of bounded linear maps is *not* continuous considered as a function of two variables (cf. [Di], part I, chapter 3); however, it is continuous with respect to each variable. Thus, we obtain that for any μ

$$\lim_{\lambda} g_\mu \circ f_\lambda = g_\mu \circ f$$

and so $g_\mu \circ f \in C$ for any μ . Now, using convergence with respect to the other variable, we obtain that $g_\mu \circ f$ converges to $g \circ f$ and so $g \circ f$ belongs to C .

(b) *Completion by projections.* We will say that an additive category $\mathcal{C}_{\mathcal{A}}$ of Hilbert representations of a $*$ -algebra \mathcal{A} is an *almost von Neumann category* if it satisfies condition (ii) of section 1.2 and condition (v) of section 2.1.

Our aim in this subsection is to show how any almost von Neumann category can be canonically completed to a von Neumann category.

Let $\mathcal{C}_{\mathcal{A}}$ be an almost von Neumann category. Consider a larger category $\tilde{\mathcal{C}}_{\mathcal{A}}$ of $*$ -representations of \mathcal{A} whose objects are the Hilbert representations of the form $e\mathcal{H}$, where $\mathcal{H} \in \text{ob}(\mathcal{C}_{\mathcal{A}})$ and $e \in \text{Hom}_{\mathcal{C}_{\mathcal{A}}}(\mathcal{H}, \mathcal{H})$ is a projector $e^* = e$, $e^2 = e$. If $e_1\mathcal{H}_1$ and $e_2\mathcal{H}_2$ are two objects of $\tilde{\mathcal{C}}_{\mathcal{A}}$ then a *morphism*

$$f : e_1\mathcal{H}_1 \rightarrow e_2\mathcal{H}_2$$

is defined as a bounded linear map f between the Hilbert spaces $e_1\mathcal{H}_1$ and $e_2\mathcal{H}_2$ having the form $f = e_2ge_1$ for some $g \in \text{Hom}_{\mathcal{C}_{\mathcal{A}}}(\mathcal{H}_1, \mathcal{H}_2)$. Note, that two different maps $g_1, g_2 \in \text{Hom}_{\mathcal{C}_{\mathcal{A}}}(\mathcal{H}_1, \mathcal{H}_2)$ may determine the same map $f = e_2g_1e_1 = e_2g_2e_1$; in this case they are considered as defining the same morphism in $\tilde{\mathcal{C}}_{\mathcal{A}}$.

It is clear that $\tilde{\mathcal{C}}_{\mathcal{A}}$ is a von Neumann category. For example, the adjoint of a morphism $f = e_2ge_1$ is $f^* = e_1g^*e_2$; thus condition (ii) is satisfied. The kernel of a morphism $f : e_1\mathcal{H}_1 \rightarrow e_2\mathcal{H}_2$, where $f = e_2ge_1$, is $e\mathcal{H}_1$, with e denoting $e_1 \wedge e'_1$. Here e'_1 is the projection onto the kernel of g and the wedge denotes the greatest lower bound of e_1 and e'_1 . Note that both e'_1 and e belong to $\text{Hom}_{\mathcal{C}_{\mathcal{A}}}(\mathcal{H}_1, \mathcal{H}_1)$ by [Di], part I, chapter 1, §2, and by [SZ], 3.7.

There is the obvious embedding $\mathcal{C}_{\mathcal{A}} \rightarrow \tilde{\mathcal{C}}_{\mathcal{A}}$ which sends \mathcal{H} to $1 \cdot \mathcal{H}$. Thus we may identify $\mathcal{C}_{\mathcal{A}}$ with a full subcategory of $\tilde{\mathcal{C}}_{\mathcal{A}}$.

2.6. Examples of von Neumann categories.

Example 1. Let \mathcal{A} be an arbitrary $*$ -algebra and let $\mathcal{C}_{\mathcal{A}}$ be the category of all Hilbert representations of \mathcal{A} . The objects of this category are all Hilbert representations of \mathcal{A} and the set of morphisms between two given representations comprise all bounded linear maps commuting with the \mathcal{A} action. Then $\mathcal{C}_{\mathcal{A}}$ is a von Neumann category. It is actually the largest possible von Neumann category of representations of \mathcal{A} . One may also consider the full subcategory of $\mathcal{C}_{\mathcal{A}}$ generated by finite objects; it is again a von Neumann category.

Example 2. The construction which we describe here is a source of many important examples. Some of them will be discussed below.

Let $\mathcal{C}_{\mathcal{A}}$ be a von Neumann category and let \mathcal{B} be a von Neumann algebra acting on a separable Hilbert space \mathcal{H} . Denote by \mathcal{B}' the commutant of \mathcal{B} .

Starting from these data we will first describe a category of Hilbert representations of the $*$ -algebra $\mathcal{A} \otimes \mathcal{B}'$ which we denote $\mathcal{C}_{\mathcal{A} \otimes \mathcal{B}'}$.

The objects of category $\mathcal{C}_{\mathcal{A} \otimes \mathcal{B}'}$ are the *Hilbert (completed) tensor products* of the form $\mathcal{H}_1 \otimes \mathcal{H}$ where $\mathcal{H}_1 \in \text{ob}(\mathcal{C}_{\mathcal{A}})$ with the obvious action of the algebra $\mathcal{A} \otimes \mathcal{B}'$ (here, abusing the notations, the same sign \otimes denotes the *algebraic* tensor product).

Given two objects $\mathcal{H}_1 \otimes \mathcal{H}$ and $\mathcal{H}_2 \otimes \mathcal{H}$ of $\mathcal{C}_{\mathcal{A} \otimes \mathcal{B}'}$, the set of morphisms

$$\text{Hom}_{\mathcal{C}_{\mathcal{A} \otimes \mathcal{B}'}}(\mathcal{H}_1 \otimes \mathcal{H}, \mathcal{H}_2 \otimes \mathcal{H}) \quad (2-1)$$

is defined as the set of all bounded linear maps $\mathcal{H}_1 \otimes \mathcal{H} \rightarrow \mathcal{H}_2 \otimes \mathcal{H}$ of the form

$$\sum_i f_i \otimes g_i \quad (2-2)$$

where the sum is finite and $f_i \in \text{Hom}_{\mathcal{C}_{\mathcal{A}}}(\mathcal{H}_1, \mathcal{H}_2)$ and $g_i \in \mathcal{B}$.

The obtained category $\mathcal{C}_{\mathcal{A} \otimes \mathcal{B}'}$ is self-adjoint (i.e. it satisfies condition (ii) of section 1.2) but *it may be not a von Neumann category* since it may not satisfy (i) of section 1.2 and (v) of section 2.1.

To make a von Neumann category out of $\mathcal{C}_{\mathcal{A} \otimes \mathcal{B}'}$ we will apply two completion constructions described in 2.5. Namely, we will first apply the completion construction of 2.5.(a) completing the set of morphisms; then we will apply the construction of 2.5.(b) completing by projectors. The resulting category will be denoted

$$\mathcal{C}_{\mathcal{A}} \otimes \mathcal{H}. \quad (2-3)$$

It is a von Neumann category of representations of $\mathcal{A} \otimes \mathcal{B}'$.

Example 3. Here we will consider a special case of the previous example. We emphasize it because it appears in many applications. It is actually simpler than the general case since here we do not need to perform *completing the set of morphisms*, section 2.5(a).

Let \mathcal{B} be a von Neumann algebra acting on a Hilbert space \mathcal{H} . Denote by \mathcal{B}' be the commutant of \mathcal{B} . Consider example 2 above with $\mathcal{A} = \mathbb{C}$ and with $\mathcal{C}_{\mathcal{A}}$ being the category of finite dimensional Hilbert spaces. Then the category $\mathcal{C}_{\mathcal{A}} \otimes \mathcal{H}$ is a von Neumann category of Hilbert representations of \mathcal{B}' .

Objects of category $\mathcal{C}_{\mathcal{A}} \otimes \mathcal{H}$ are in one-to-one correspondence with projections $e \in M(n) \otimes \mathcal{B}$, $e^* = e$, $e^2 = e$, for some n , where $M(n)$ is the $n \times n$ -matrix algebra. For each projection e the corresponding Hilbert representation of \mathcal{B}' is $e(\mathbb{C}^n \otimes \mathcal{H})$. If e_1 and e_2 are two projections, then the set of morphisms $\text{Hom}_{\mathcal{C}_{\mathcal{A}} \otimes \mathcal{H}}(e_1(\mathbb{C}^{n_1} \otimes \mathcal{H}), e_2(\mathbb{C}^{n_2} \otimes \mathcal{H}))$ is the set of all bounded linear maps of the form $e_2 b e_1 : e_1(\mathbb{C}^{n_1} \otimes \mathcal{H}) \rightarrow e_2(\mathbb{C}^{n_2} \otimes \mathcal{H})$ where b is given by an $n_1 \times n_2$ -matrix with entries in \mathcal{B} .

Studying this von Neumann category $\mathcal{C}_{\mathcal{A}} \otimes \mathcal{H}$ is in some sense equivalent to studying the original von Neumann algebra \mathcal{B} . In particular, we see that this von Neumann category $\mathcal{C}_{\mathcal{A}} \otimes \mathcal{H}$ is finite if and only if the original von Neumann algebra \mathcal{B} is finite.

Example 4. The following example corresponds to the extended category constructed in [Fa, Fa1].

Let \mathcal{B} be a finite von Neumann algebra supplied with a finite, normal and faithful trace $\tau : \mathcal{B} \rightarrow \mathbb{C}$. Consider the completion $\ell_{\tau}^2(\mathcal{B})$ of \mathcal{B} with respect to the scalar product $\langle a, b \rangle = \tau(ab^*)$. Then we have \mathcal{B} acting on the Hilbert space $\mathcal{H} = \ell_{\tau}^2(\mathcal{B})$ from the left and from the right, and these actions commute. In fact, to make it compatible with our previous constructions, we may think of the left action of \mathcal{B} on \mathcal{H} and identify the commutant \mathcal{B}' with \mathcal{B}^{\bullet} , the opposite algebra of \mathcal{B} .

Now, if $\mathcal{C}_{\mathcal{A}}$ denotes the von Neumann category of finite dimensional Euclidean spaces (here $\mathcal{A} = \mathbb{C}$), then we may consider the von Neumann category $\mathcal{C}_{\mathcal{A}} \otimes \mathcal{H}$ as explained in Example 2. It is a von Neumann category of $*$ -representations of the von Neumann algebra \mathcal{B}^{\bullet} .

Example 5. Here we consider the special case of example 4 which is most important for topological applications.

Let π be a discrete group and let $\mathcal{H} = \ell^2(\pi)$ denote the Hilbert completion of the group ring $\mathbb{C}[\pi]$. The group ring $\mathbb{C}[\pi]$ acts on \mathcal{H} from both sides. We denote by $\mathcal{B} = \mathcal{N}(\pi)$ the von Neumann algebra of π which consists of all bounded linear maps $T : \mathcal{H} \rightarrow \mathcal{H}$ commuting with the left action of $\mathbb{C}[\pi]$. The commutant of $\mathcal{N}(\pi)$ can be identified with the opposite algebra $\mathcal{N}(\pi)^{\bullet}$.

As above, let $\mathcal{C}_{\mathcal{A}}$ denote the von Neumann category of finite dimensional Euclidean spaces, then we may form the von Neumann category $\mathcal{C}_{\mathcal{A}} \otimes \mathcal{H}$ as in Example 2. It is a von Neumann category of $*$ -representations of $\mathcal{N}(\pi)^{\bullet}$.

Example 6. A slightly more general example can be obtained as follows. Let V be a Hilbert representation of π (possibly, finite dimensional) and let $\mathcal{H} = V \otimes \ell^2(\pi)$ (the completed tensor product). Then we have two actions of π on \mathcal{H} : it acts diagonally from the left and it acts from the right through the second factor $\ell^2(\pi)$. These two actions commute. The right action clearly extends to an action of the von Neumann algebra $\mathcal{N}(\pi)^{\bullet}$. The left action may also be extended (by taking the weak closure) to action of a von Neumann algebra, say \mathcal{B} , containing the group ring $\mathbb{C}[\pi]$. Now we may apply construction of example 3; the obtained von Neumann category will be a category of $*$ -representations of the algebra $\mathcal{N}(\pi)^{\bullet}$.

Example 7. *Fields of Hilbert spaces.* This example is important for studying families.

Let Z be a locally compact Hausdorff space and let μ be a positive Radon measure on Z . Let \mathcal{A} denote the algebra $L_{\mathbb{C}}^{\infty}(Z, \mu)$ (the space of essentially bounded μ -measurable complex valued functions on Z , in which two functions equal locally almost everywhere, are identical). The involution on \mathcal{A} is given by the complex conjugation.

We will construct a category of Hilbert representations of \mathcal{A} as follows. The objects of $\mathcal{C}_{\mathcal{A}}$ are in one-to-one correspondence with the μ -measurable fields of finite-dimensional Hilbert spaces $\xi \rightarrow \mathcal{H}(\xi)$ over (Z, μ) , cf. [Di], part II, chapter 1. For any such measurable field of Hilbert spaces, the corresponding Hilbert space is the direct integral

$$\mathcal{H} = \int^{\oplus} \mathcal{H}(\xi) d\mu(\xi) \quad (2-4)$$

defined as in [Di], part II, chapter 1. The algebra \mathcal{A} acts on the Hilbert space (2-4) by pointwise multiplication.

Suppose that we have two μ -measurable finite-dimensional fields of Hilbert spaces $\xi \rightarrow \mathcal{H}(\xi)$ and $\xi \rightarrow \mathcal{H}'(\xi)$ over Z . Then we have two corresponding Hilbert spaces, \mathcal{H} and \mathcal{H}' , given as direct integrals (2-4). We define the set of morphisms $\text{Hom}_{\mathcal{C}_{\mathcal{A}}}(\mathcal{H}, \mathcal{H}')$ as the set of all bounded linear maps $\mathcal{H} \rightarrow \mathcal{H}'$ given by *decomposable linear maps*

$$T = \int^{\oplus} T(\xi) d\mu(\xi), \quad (2-5)$$

where $T(\xi)$ is an *essentially bounded measurable field of linear maps* $T(\xi) : \mathcal{H}(\xi) \rightarrow \mathcal{H}'(\xi)$, cf. [Di], part II, chapter 2.

The kernel of any decomposable linear map as above can be represented as the direct integral of a finite-dimensional field of Hilbert spaces and so the condition (i) of section 1.2 is satisfied. The condition (ii) of section 1.2 is also satisfied since the adjoint of the map T given by (2-5) is

$$T^* = \int^{\oplus} T(\xi)^* d\mu(\xi)$$

(by [Di], part II, chapter 2, §3, Proposition 3). The condition (v) of section 2.1 is satisfied as follows from Theorem 1 of [Di], part II, chapter 2, §5.

Thus, we obtain a von Neumann category of Hilbert representations of the algebra $\mathcal{A} = L^\infty_{\mathbb{C}}(Z, \mu)$.

This category is finite.

Note that this category $\mathcal{C}_{\mathcal{A}}$ depends only on the *class of the measure* μ .

The category $\mathcal{C}_{\mathcal{A}}$ of this example contains as a full subcategory the category $\mathcal{C}'_{\mathcal{A}}$ which is obtained by the construction of example 3, applied to the Hilbert space $\mathcal{H} = L^2(Z, \mu)$ and the von Neumann algebra $\mathcal{B} = L^\infty_{\mathbb{C}}(Z, \mu)$ acting on it. Then the commutant \mathcal{B}' is again $L^\infty_{\mathbb{C}}(Z, \mu)$. In fact, the objects of the smaller category $\mathcal{C}'_{\mathcal{A}}$ can be considered as the measurable fields of finite dimensional Hilbert spaces $\xi \rightarrow \mathcal{H}(\xi)$ over Z such that the dimension of $\mathcal{H}(\xi)$ is essentially bounded.

Example 8. We can combine the above examples as follows. For any von Neumann category $\mathcal{C}_{\mathcal{A}}$ and for any locally compact Hausdorff space Z with a positive Radon measure μ , we have a von Neumann category $\mathcal{C}_{\mathcal{A}} \otimes L^2(Z, \mu)$ constructed as in example 2. It is a category of Hilbert representations of the algebra $\mathcal{A} \otimes L^\infty(Z, \mu)$. The objects of this category may be thought of as "*measurable fields of objects of the category $\mathcal{C}_{\mathcal{A}}$ over the space Z , having bounded size*", generalizing the previous example 7.

Example 9. Filtrations. Consider the algebra \mathcal{A} which has infinite number of generators e_i , where $i = 1, 2, \dots$, satisfying the relations

$$e_i e_j = e_j e_i = e_{\min\{i, j\}} \quad \text{for all } i, j = 1, 2, 3, \dots \quad (2-6)$$

We define involution on \mathcal{A} by $e_i^* = e_i$.

A Hilbertian representation of \mathcal{A} consists of a Hilbert space \mathcal{H} and a filtration

$$\mathcal{H}_1 \subset \mathcal{H}_2 \subset \mathcal{H}_3 \subset \dots \subset \mathcal{H} \quad (2-7)$$

by closed subspaces $\mathcal{H}_i = e_i \mathcal{H}$.

We denote by $\mathcal{C}_{\mathcal{A}}$ the category whose objects are Hilbert representations of \mathcal{A} such that all subspaces \mathcal{H}_i are finite dimensional and $\mathcal{H} = \cup \mathcal{H}_i$. If \mathcal{H} and \mathcal{H}' are two such representations, a morphism $\mathcal{H} \rightarrow \mathcal{H}'$ in $\mathcal{C}_{\mathcal{A}}$ is defined as a bounded linear map commuting with the action of the algebra \mathcal{A} .

Observe that if \mathcal{H} is an object of $\mathcal{C}_{\mathcal{A}}$, then there is the dual decreasing filtration

$$\mathcal{H} \supset \mathcal{H}_1^\perp \supset \mathcal{H}_2^\perp \supset \dots, \quad (2-8)$$

where $\mathcal{H}_i^\perp = (1 - e_i)\mathcal{H}$ is the orthogonal complement of \mathcal{H}_i . A bounded linear map $f : \mathcal{H} \rightarrow \mathcal{H}'$ is a morphism of $\mathcal{C}_{\mathcal{A}}$ if and only if it preserves the filtration $f(\mathcal{H}_i) \subset \mathcal{H}'_i$ and the dual filtration $f(\mathcal{H}_i^\perp) \subset \mathcal{H}'_i{}^\perp$.

It is easy to check that $\mathcal{C}_{\mathcal{A}}$ is a *finite von Neumann category*.

Remark. If $\mathcal{C}_{\mathcal{A}}$ is a von Neumann algebra and if $\mathcal{B} \subset \mathcal{A}$ is a $*$ -subalgebra then $\mathcal{C}_{\mathcal{A}}$ can be obviously viewed as a von Neumann category of Hilbertian representations of \mathcal{B} .

More generally, if $\phi : \mathcal{B} \rightarrow \mathcal{A}$ is a morphism of algebras with involution then any von Neumann category $\mathcal{C}_{\mathcal{A}}$ of Hilbert representations of \mathcal{A} can be viewed (via ϕ) as a von Neumann category of Hilbert representations of \mathcal{B} .

Any full additive subcategory of a von Neumann category is again a von Neumann category. Using this remark one may construct other examples of von Neumann categories starting from the examples mentioned above.

We refer also to [GLR] for other examples.

2.7. Traces on von Neumann categories. Let $\mathcal{C}_{\mathcal{A}}$ be a von Neumann category.

Definition. A trace on $\mathcal{C}_{\mathcal{A}}$ is a function, denoted tr , which assigns to each object $\mathcal{H} \in \text{ob}(\mathcal{C}_{\mathcal{A}})$ a finite, non-negative trace

$$\text{tr}_{\mathcal{H}} : \text{Hom}_{\mathcal{C}_{\mathcal{A}}}(\mathcal{H}, \mathcal{H}) \rightarrow \mathbb{C}$$

on the von Neumann algebra $\text{Hom}_{\mathcal{C}_{\mathcal{A}}}(\mathcal{H}, \mathcal{H})$; in other words $\text{tr}_{\mathcal{H}}$ assumes (finite) values in \mathbb{C} , $\text{tr}_{\mathcal{H}}(a)$ is non-negative on non-negative elements a of $\text{Hom}_{\mathcal{C}_{\mathcal{A}}}(\mathcal{H}, \mathcal{H})$, and $\text{tr}_{\mathcal{H}}$ is traceful, i.e. $\text{tr}_{\mathcal{H}}(ab) = \text{tr}_{\mathcal{H}}(ba)$, for $a, b \in \text{Hom}_{\mathcal{C}_{\mathcal{A}}}(\mathcal{H}, \mathcal{H})$.

It is also assumed that for any pair of representations \mathcal{H}_1 and \mathcal{H}_2 the corresponding traces $\text{tr}_{\mathcal{H}_1}$, $\text{tr}_{\mathcal{H}_2}$ and $\text{tr}_{\mathcal{H}_1 \oplus \mathcal{H}_2}$ are related as follows: if $f \in \text{Hom}_{\mathcal{C}_{\mathcal{A}}}(\mathcal{H}_1 \oplus \mathcal{H}_2, \mathcal{H}_1 \oplus \mathcal{H}_2)$ is given by a 2×2 matrix (f_{ij}) , where $f_{ij} : \mathcal{H}_i \rightarrow \mathcal{H}_j$, $i, j = 1, 2$, then

$$\text{tr}_{\mathcal{H}_1 \oplus \mathcal{H}_2}(f) = \text{tr}_{\mathcal{H}_1}(f_{11}) + \text{tr}_{\mathcal{H}_2}(f_{22}). \quad (2-9)$$

We will say that a trace tr on a von Neumann category is *normal* (or *faithful*) iff for each non-zero $\mathcal{H} \in \text{ob}(\mathcal{C}_{\mathcal{A}})$ the trace $\text{tr}_{\mathcal{H}}$ on the von Neumann algebra $\text{Hom}_{\mathcal{C}_{\mathcal{A}}}(\mathcal{H}, \mathcal{H})$ is normal (faithful).

I am going to show elsewhere that non-normal traces produce extremally interesting numerical invariants of projective and torsion objects of extended abelian categories, which have important geometric applications. However in this paper I will consider only normal traces.

Examples of traces. Consider the von Neumann category $\mathcal{C}_{\mathcal{A}} \otimes \mathcal{H}$ of Example 3 of section 2.6. Then any finite trace on the von Neumann algebra \mathcal{B} defines a trace on the corresponding von Neumann category $\mathcal{C}_{\mathcal{A}} \otimes \mathcal{H}$. Namely, given a morphism $e_2 b e_1 : e_1(\mathbb{C}^n \otimes \mathcal{H}) \rightarrow e_1(\mathbb{C}^n \otimes \mathcal{H})$, where $e_1, e_2, b \in M(n) \otimes \mathcal{B}$ and e_1 and e_2 are projectors (cf. section 2.6, example 3) we may consider it as an $n \times n$ -matrix (b_{ij}) with entries in \mathcal{B} and so we may define

$$\text{tr}(e_2 b e_1) = \sum_{i=1}^n \text{tr}(b_{ii}). \quad (2-10)$$

Examples 4, 5, 6 of section 2.6 are special cases of example 3 and so using the previous remark we know the traces in these von Neumann categories.

It seems that von Neumann categories of examples 7 and 9 are too large to have traces. But some of their subcategories clearly have traces.

For instance, given a locally compact Hausdorff space with a positive Radon measure μ one may consider the category of measurable fields of finite dimensional Hilbert spaces $\mathcal{H}(\xi)$ over Z such that the dimensions of $\mathcal{H}(\xi)$ are essentially bounded. This category has traces. Let ν be a positive measure on Z which is absolutely continuous with respect to μ and such that $\nu(Z) < \infty$. Then ν determines the following trace on this von Neumann category

$$\text{tr}_{\nu}(T) = \int_Z \text{Tr}(T(\xi)) d\nu \quad (2-11)$$

where T is a morphism given by formula (2-5) and Tr denotes the usual finite dimensional trace.

Let us now describe a von Neumann category which is a full subcategory of the category of example 9, which admits a trace. Fix a sequence of non-negative real numbers $\mu = (\mu_n)$, such that the series $\sum \mu_n$ converges. Let \mathcal{A} denote the algebra with involution as in example 9. We will consider the category $\mathcal{C}_{\mathcal{A}}(\mu)$ of all Hilbert representations \mathcal{H} of \mathcal{A} having the following properties:

- (1) the subspaces $e_i \mathcal{H} = \mathcal{H}_i$ are all finite dimensional;
- (2) $\cup \mathcal{H}_i = \mathcal{H}$;
- (3) the series $\sum \mu_n d_n$ converges; here d_n denotes the dimension of the space $\mathcal{H}_{n-1}^\perp \cap \mathcal{H}_n$.

Morphisms of $\mathcal{C}_{\mathcal{A}}(\mu)$ are all bounded linear maps commuting with \mathcal{A} .

Now we may describe a trace in $\mathcal{C}_{\mathcal{A}}(\mu)$. Given an object $\mathcal{H} \in \text{ob}(\mathcal{C}_{\mathcal{A}}(\mu))$, and a morphism $f : \mathcal{H} \rightarrow \mathcal{H}$, define

$$\text{tr}(f) = \sum_{n=1}^{\infty} \mu_n \text{Tr}(f_n) \quad (2-12)$$

where f_n is one of the maps between finite dimensional spaces, $f_n : \mathcal{H}_{n-1}^\perp \cap \mathcal{H}_n \rightarrow \mathcal{H}_{n-1}^\perp \cap \mathcal{H}_n$, determined by f ; in other words $f_n = (e_n - e_{n-1})f(e_n - e_{n-1})$. The symbol Tr in the RHS of (2-12) is the usual finite dimensional trace. Note that the series in (2-12) converges for any bounded f . It is easy to check that (2-12) is a trace on the category $\mathcal{C}_{\mathcal{A}}(\mu)$.

Note that the notion of trace on a category is also helpful while studying representations of finite dimensional algebras. If \mathcal{A} is a finite dimensional algebra then the category $\mathcal{C}_{\mathcal{A}}$ of all finite dimensional representations of \mathcal{A} is a finite von Neumann category. The traces on this category can be described as follows. Let $\mathcal{H}_1, \mathcal{H}_2, \dots, \mathcal{H}_n$ be all different irreducible representations of \mathcal{A} . Fix positive numbers μ_1, \dots, μ_n . For any $\mathcal{H} \in \text{ob}(\mathcal{C}_{\mathcal{A}})$ and an \mathcal{A} -homomorphism $f : \mathcal{H} \rightarrow \mathcal{H}$ define

$$\text{tr}_{\mathcal{H}}(f) = \sum_{i=1}^n \mu_i \text{tr}(f_i)$$

where f_i denotes the linear map $\text{Hom}_{\mathcal{A}}(\mathcal{H}_i, \mathcal{H}) \rightarrow \text{Hom}_{\mathcal{A}}(\mathcal{H}_i, \mathcal{H})$ induced by f and the symbol tr on the right denotes the usual finite dimensional trace (which is up to normalization the only trace in the category of finite dimensional vector spaces).

Note also that it is sometimes more convenient to abandon the condition of positivity of traces on categories and consider more general traces than allowed by Definition 2.7.

2.8. Von Neumann dimension. Given a trace tr on a finite von Neumann category, one may measure the sizes of objects of $\mathcal{C}_{\mathcal{A}}$ by *their von Neumann dimension*

$$\dim \mathcal{H} = \dim_{\text{tr}}(\mathcal{H}) = \text{tr}_{\mathcal{H}}(\text{id}_{\mathcal{H}}) \quad (2-13)$$

with respect to the chosen trace. The dimensions of non-zero objects of $\mathcal{C}_{\mathcal{A}}$ are strictly positive if and only if the trace tr is faithful.

A von Neumann category admitting a faithful trace is necessarily finite.

2.9. Lemma. *For any pair of morphisms $f : \mathcal{H}_1 \rightarrow \mathcal{H}_2$ and $g : \mathcal{H}_2 \rightarrow \mathcal{H}_1$ of a von Neumann category \mathcal{C}_A supplied with a trace tr holds*

$$\text{tr}_{\mathcal{H}_2}(fg) = \text{tr}_{\mathcal{H}_1}(gf). \quad (2-14)$$

Proof.

$$\begin{aligned} \text{tr}_{\mathcal{H}_1}(gf) &= \text{tr}_{\mathcal{H}_1 \oplus \mathcal{H}_2} \left(\begin{bmatrix} gf & 0 \\ 0 & 0 \end{bmatrix} \right) \\ &= \text{tr}_{\mathcal{H}_1 \oplus \mathcal{H}_2} \left(\begin{bmatrix} 0 & 0 \\ g & 0 \end{bmatrix} \circ \begin{bmatrix} 0 & f \\ 0 & 0 \end{bmatrix} \right) \\ &= \text{tr}_{\mathcal{H}_1 \oplus \mathcal{H}_2} \left(\begin{bmatrix} 0 & f \\ 0 & 0 \end{bmatrix} \circ \begin{bmatrix} 0 & 0 \\ g & 0 \end{bmatrix} \right) \\ &= \text{tr}_{\mathcal{H}_1 \oplus \mathcal{H}_2} \left(\begin{bmatrix} 0 & 0 \\ 0 & fg \end{bmatrix} \right) \\ &= \text{tr}_{\mathcal{H}_2}(fg) \end{aligned}$$

2.10. Corollary. *If $\mathcal{H}_1, \mathcal{H}_2 \in \text{ob}(\mathcal{C}_A)$ are isomorphic in \mathcal{C}_A then their von Neumann dimensions are equal $\dim(\mathcal{H}_1) = \dim(\mathcal{H}_2)$*

Proof. Let $f : \mathcal{H}_1 \rightarrow \mathcal{H}_2$ and $g : \mathcal{H}_2 \rightarrow \mathcal{H}_1$ be mutually inverse isomorphisms. Then

$$\dim \mathcal{H}_1 = \text{tr}_{\mathcal{H}_1}(\text{id}_{\mathcal{H}_1}) = \text{tr}_{\mathcal{H}_1}(gf) = \text{tr}_{\mathcal{H}_2}(fg) = \text{tr}_{\mathcal{H}_2}(\text{id}_{\mathcal{H}_2}) = \dim \mathcal{H}_2$$

It is well known that any finite von Neumann algebra admits a finite normal faithful trace. It is probably not true that any finite von Neumann category admits a faithful trace.

2.11. Question. *Which conditions guarantee existence of faithful traces on a given finite von Neumann category?*

§3. Torsion objects of the extended category

In this section we will further study the construction of the extended abelian category $\mathcal{E}(\mathcal{C}_A)$ of section 1.

We will here assume everywhere in this section that *the original category \mathcal{C}_A is a finite von Neumann category.*

3.1. Definition. An object $\mathcal{X} = (\alpha : A' \rightarrow A)$ of the extended category $\mathcal{E}(\mathcal{C}_A)$ is called *torsion* iff the image of α is dense in A .

Equivalently, an object \mathcal{X} of $\mathcal{E}(\mathcal{C}_A)$ is torsion if and only if it admits no nontrivial homomorphisms in projective objects of $\mathcal{E}(\mathcal{C}_A)$.

We will denote by $\mathcal{T}(\mathcal{C}_A)$ the full subcategory of $\mathcal{E}(\mathcal{C}_A)$ generated by all torsion objects. $\mathcal{T}(\mathcal{C}_A)$ is called *the torsion subcategory of $\mathcal{E}(\mathcal{C}_A)$.*

Note, that in general (without assuming that \mathcal{C}_A is a finite von Neumann category) it is possible that a projective object is subobject of a torsion object. Our desire to avoid this situation explains the assumption on \mathcal{C}_A .

3.2 Proposition. *Given an exact sequence*

$$0 \rightarrow \mathcal{X}' \rightarrow \mathcal{X} \rightarrow \mathcal{X}'' \rightarrow 0 \quad (3-1)$$

of objects and morphisms of $\mathcal{E}(\mathcal{C}_A)$, where \mathcal{C}_A is a finite von Neumann category, the middle object \mathcal{X} is torsion if and only if both \mathcal{X}' and \mathcal{X}'' are torsion.

Proof. The only nontrivial part of the proof consists in showing that a subobject of a torsion object is necessarily torsion. As we will see, this follows from Proposition 2.4.

Suppose that the diagram

$$\begin{array}{ccc} (\alpha : A' & \longrightarrow & A) \\ & & \downarrow f \\ (\beta : B' & \longrightarrow & B) \end{array}$$

represents a monomorphism in $\mathcal{E}(\mathcal{C}_A)$ and $(\beta : B' \rightarrow B)$ is torsion. Then from Proposition 1.6 it follows that

$$\alpha(A') \supset f^{-1}(\beta(B')). \quad (3-2)$$

Suppose that $(\alpha : A' \rightarrow A)$ is not torsion. Then there is a nontrivial closed submodule $C \subset A$ such that $\text{cl}(\text{im}(\alpha)) \cap C = 0$. We obtain from (3-2) that f maps $C \rightarrow B$ monomorphically and that $f(C) \cap \beta(B') = 0$. This contradicts Proposition 2.4. \square

3.3. Corollary. *The torsion subcategory $\mathcal{T}(\mathcal{C}_A)$ is an abelian subcategory of $\mathcal{E}(\mathcal{C}_A)$.*

Recall that we assume that \mathcal{C}_A is a finite von Neumann category.

3.4. Functor of the torsion part. Given an arbitrary object $\mathcal{X} = (\alpha : A' \rightarrow A)$ of $\mathcal{E}(\mathcal{C}_A)$ one may consider the following torsion object

$$\mathcal{T}(\mathcal{X}) = (\alpha : A' \rightarrow \text{cl}(\text{im}(\alpha))) \quad (3-3)$$

which is called *the torsion part of \mathcal{X}* . There is an obvious morphism $\mathcal{T}(\mathcal{X}) \rightarrow \mathcal{X}$ which is a monomorphism and so $\mathcal{T}(\mathcal{X})$ can be viewed as a subobject of \mathcal{X} . It is clear that any morphism $\mathcal{X} \rightarrow \mathcal{Y}$ in $\mathcal{E}(\mathcal{C}_A)$ maps the torsion part of \mathcal{X} into the torsion part of \mathcal{Y} . Thus, \mathcal{T} is naturally defined as a functor

$$\mathcal{T} : \mathcal{E}(\mathcal{C}_A) \rightarrow \mathcal{C}_A. \quad (3-4)$$

Using Proposition 1.6, we easily see that, the following sequence is exact

$$0 \rightarrow \mathcal{T}(\mathcal{X}) \rightarrow \mathcal{X} \rightarrow P(\mathcal{X}) \rightarrow 0. \quad (3-5)$$

Here P denote the functor of the projective part which was discussed in 1.11. Since $P(\mathcal{X})$ is projective (by Proposition 1.9) the sequence (3-5) splits and so we have

$$\mathcal{X} = \mathcal{T}(\mathcal{X}) \oplus P(\mathcal{X}). \quad (3-6)$$

We obtain: *isomorphism type of an object of the extended category $\mathcal{E}(\mathcal{C}_A)$ is determined by the isomorphism types of its projective and torsion parts.*

Note that representation (3-6) is not canonical.

3.5. Our next aim is to apply the construction of S.Novikov and M.Shubin [NS] which allows to measure the sizes of torsion objects of the extended category.

We start with the following technical statement which will be quite important for the sequel.

We will say that two torsion objects $\mathcal{X} = (\alpha : A' \rightarrow A)$ and $\mathcal{Y} = (\beta : B' \rightarrow B)$ of the extended category $\mathcal{E}(\mathcal{C}_A)$ (with injective α and β) are *strongly isomorphic* if there exist two isomorphisms $f : A \rightarrow B$ and $g : A' \rightarrow B'$ in \mathcal{C}_A such that $f \circ \alpha = \beta \circ g$. Note that strongly isomorphic objects are necessarily isomorphic, but the converse is not true.

3.6. Proposition. *Let $\mathcal{X} = (\alpha : A' \rightarrow A)$ and $\mathcal{Y} = (\beta : B' \rightarrow B)$ be two isomorphic torsion objects of the extended category $\mathcal{E}(\mathcal{C}_A)$ with injective α and β . Then it is possible to perform a finite number of excisions on \mathcal{X} and on \mathcal{Y} (at most four!) such that the obtained objects will be strongly isomorphic.*

Proof. Suppose that the diagram

$$\begin{array}{ccc} (\alpha : A' & \longrightarrow & A) \\ g \downarrow & & \downarrow f \\ (\beta : B' & \longrightarrow & B) \end{array} \quad (3-7)$$

represents an isomorphism in $\mathcal{E}(\mathcal{C}_A)$. From Proposition 1.6 it follows (using the fact that the morphism represented by f is a monomorphism) that $\alpha(A') \supset f^{-1}(\beta(B'))$. In particular, the kernel of f is contained in the image of α . Thus, we may perform an excision with respect to $\alpha^{-1}(\ker(f))$. As a result we will obtain a diagram similar to (3-7), which represents an isomorphism in $\mathcal{E}(\mathcal{C}_A)$ with f and g injective.

Consider now $\text{im}(f)^\perp \subset B$ and the orthogonal projection $\pi : B \rightarrow \text{im}(f)^\perp$. From Proposition 1.6 it follows (using the fact that the morphism represented by f is an epimorphism) that $\text{im}(f)^\perp$ coincides with the image of $\pi \circ \beta$. Thus we may perform an excision on $(\pi \circ \beta)^{-1}(\text{im}(f)^\perp)$ such that the initial isomorphism will be represented by a diagram of form (3-7) such that f and g are injective with dense images.

By Proposition 2.3 all Hilbert representations A, B, A', B' are isomorphic in \mathcal{C}_A and so we may identify them. So, we may assume that we are given an isomorphism in $\mathcal{E}(\mathcal{C}_A)$ of the form

$$\begin{array}{ccc} (\alpha : A & \longrightarrow & A) \\ g \downarrow & & \downarrow f \\ (\beta : A & \longrightarrow & A) \end{array} \quad (3-8)$$

with α, β, f and g being injective with dense images. Also, we may suppose that α and β are self-adjoint and positive.

By Proposition 1.6, the sequence

$$0 \rightarrow A \xrightarrow{\alpha \oplus -g} A \oplus A \xrightarrow{(f, \beta)} A \rightarrow 0$$

is exact in \mathcal{C}_A and so it splits. Thus there exist morphisms

$$\sigma, \delta : A \rightarrow A$$

such that

$$\text{id}_A = \sigma\alpha + \delta g. \quad (3-9)$$

Choose $\epsilon > 0$ such that $\epsilon \cdot \|\sigma\| < 1$.

Now, we may find a splitting

$$A = P \oplus Q$$

such that $\alpha(P) = P$ and $\alpha(Q) \subset Q$ and the norm of the restriction of α on Q will be less than ϵ . This may be achieved by choosing a corresponding spectral projector from the spectral decomposition of α . Then from equation (3-9) we find that $\pi_Q \circ \delta g|_Q : Q \rightarrow Q$ is an isomorphism (where $\pi_Q : A \rightarrow Q$ denotes the orthogonal projection) and thus $g|_Q : Q \rightarrow A$ has closed image. Hence we obtain that the initial isomorphism will be represented by a diagram

$$\begin{array}{ccc} (\alpha|_Q : Q & \longrightarrow & Q) \\ g|_Q \downarrow & & \downarrow f|_Q \\ (\beta : A & \longrightarrow & A) \end{array} \quad (3-10)$$

such that the image of $g|_Q$ is closed. Then the image of $f|_Q$ is closed (here one again uses Proposition 1.6 and the fact that the diagram (3-10) represents an isomorphism). As above, we may perform an excision with respect to the orthogonal complement to $f(Q)$ in A to obtain a strong isomorphism. \square

3.7. The spectral density function. We are now ready to describe the construction which was used by S.Novikov and M.Shubin [NS] in order to measure the spectrum near zero. In the language of the present paper this construction assigns to a torsion object of the extended category a *spectral density function*, which is real valued function determined up to certain equivalence relation. There is also an important numerical invariant which can be extracted from the spectral density function (it is called *the Novikov-Shubin number*, cf. 3.9).

Suppose that \mathcal{C}_A is a *finite von Neumann category supplied with a trace* tr .

Let $\mathcal{X} = (\alpha : A' \rightarrow A)$ be a torsion object of the extended category represented by an injective morphism α . Let $T : A' \rightarrow A'$ be a positive self-adjoint operator, $T \in \text{Hom}_{\mathcal{C}_A}(A', A')$, such that $T^2 = \alpha^* \alpha$. By the spectral theorem, we have representation

$$T = \int_0^\infty \lambda dE_\lambda \quad (3-11)$$

and we denote by $F(\lambda)$ the von Neumann dimension (with respect to the chosen trace tr) of the subspace $E_\lambda A'$ determined by the spectral projector E_λ , where $\lambda > 0$:

$$F(\lambda) = \dim E_\lambda A' \quad (3-12)$$

The obtained function $F(\lambda)$ is called *the spectral density function of \mathcal{X}* .

It is clear that $F(\lambda)$ is non-decreasing and right continuous (since the trace tr is supposed to be normal) function with $F(0) = 0$. If the trace tr is faithful then $F(\lambda) > 0$ for $\lambda > 0$. Note, that in general we do not require the trace tr to be faithful.

Let us emphasize again that *the spectral density function of a torsion object \mathcal{X} depends on \mathcal{X} and on the trace tr although our notation does not indicate this fact.*

Since we are interested in getting an isomorphism invariant of \mathcal{X} , we will focus our attention only in the behavior of the spectral density function *near zero*; in fact, performing excisions, we may change the spectral density function $F(\lambda)$ arbitrarily away from zero.

Two spectral density functions $F(\lambda)$ and $G(\lambda)$ are called *dilatationally equivalent* (denoted $F(\lambda) \sim G(\lambda)$) if there exist constants $C > 1$ and $\epsilon > 0$ such that

$$G(C^{-1}\lambda) \leq F(\lambda) \leq G(C\lambda)$$

for all $\lambda \in (0, \epsilon)$.

The following statement was proven by M.Gromov and M.Shubin [GS] in a much more general situation. We include its proof for the sake of completeness.

3.8. Proposition. *Spectral density functions of isomorphic torsion objects \mathcal{X} and \mathcal{Y} are dilatationally equivalent.*

Proof. Because of Proposition 3.6, it is enough to show that the dilatational equivalence class of the spectral density function does not change under excisions and under strong isomorphisms.

Excisions clearly change the spectral density function only away from zero. Thus we need only to consider strong isomorphisms. Suppose that a commutative diagram

$$\begin{array}{ccc} (\alpha : A' & \longrightarrow & A) \\ g \downarrow & & \downarrow f \\ (\beta : B' & \longrightarrow & B) \end{array} \quad (3-14)$$

(with α and β injective) represents a strong isomorphism (i.e. f and g are isomorphisms). Let $F(\lambda)$ and $G(\lambda)$ denote the spectral density functions of $\mathcal{X} = (\alpha : A' \rightarrow A)$ and $\mathcal{Y} = (\beta : B' \rightarrow B)$ correspondingly. Since f and g are isomorphisms, there exists a constant $K > 1$ such that

$$K^{-1}|x| \leq |f(x)| \leq K|x| \quad \text{and also} \quad K^{-1}|x| \leq |g(x)| \leq K|x| \quad (3-15)$$

for all x .

We will use the variational characterization of the spectral density function given by M.Gromov and M.Shubin [GS]. Namely, for any $\lambda > 0$ consider the cone

$$C_\lambda(\mathcal{X}) = \{x \in A'; |\alpha(x)| \leq \lambda|x|\}. \quad (3-16)$$

Then

$$F(\lambda) = \sup \dim L, \quad (3-17)$$

where L runs over all $\mathcal{C}_\mathcal{A}$ -subspaces $L \subset A'$, $L \in \text{ob}(\mathcal{C}_\mathcal{A})$.

Now, using inequalities (3-15) we obtain that if $L \subset C_\lambda(\mathcal{X})$ then $gL \subset C_{K^2\lambda}(\mathcal{Y})$. Combined with (3-17), it implies $G(K^2\lambda) \geq F(\lambda)$.

Changing the role of \mathcal{X} and \mathcal{Y} , one obtains that $F(K^2\lambda) \geq G(\lambda)$. \square

3.9. The Novikov - Shubin number. In this subsection we will assume that the trace tr is normal.

Let \mathcal{X} be a torsion object of the extended category $\mathcal{E}(\mathcal{C}_{\mathcal{A}})$. Suppose that its spectral density function $F(\lambda)$ constructed with respect to a given trace tr on the initial von Neumann category $\mathcal{C}_{\mathcal{A}}$ is *positive* for $\lambda > 0$. Then one defines the following non-negative real number (which allowed to be ∞)

$$\mathfrak{ns}(\mathcal{X}) = \lim_{\lambda \rightarrow 0^+} \inf \frac{\log F(\lambda)}{\log \lambda} \in [0, \infty], \quad (3-18)$$

which is called *the Novikov - Shubin number or the Novikov - Shubin invariant of \mathcal{X}* . Sometimes we will use the notation

$$\mathfrak{ns}(\mathcal{X}) = \mathfrak{ns}_{\text{tr}}(\mathcal{X}) \quad (3-19)$$

in order to emphasize dependence on the trace tr .

From Proposition 3.8 it follows that *isomorphic torsion objects $\mathcal{X} \simeq \mathcal{Y}$ have equal Novikov - Shubin numbers $\mathfrak{ns}(\mathcal{X}) = \mathfrak{ns}(\mathcal{Y})$* .

In fact, it is more convenient to use an equivalent invariant

$$\mathfrak{c}(\mathcal{X}) = \mathfrak{ns}(\mathcal{X})^{-1} \in [0, \infty], \quad (3-20)$$

which was introduced in [Fa1]; it is called *the capacity of \mathcal{X}* . The advantage of capacity against the Novikov-Shubin number consists in the fact that it adequately describes the size of a torsion object: larger torsion objects have larger capacity, trivial object has zero capacity, there may also exist torsion objects having infinite capacity.

If the spectral density function $F(\lambda)$ vanishes for some $\lambda > 0$ then we will define $\mathfrak{c}(\mathcal{X}) = 0$. This situation may happen for a nonzero \mathcal{X} if the trace is not faithful.

The following statement describes the main properties of the Novikov - Shubin capacity.

3.10. Proposition. *For any short exact sequence of torsion objects*

$$0 \rightarrow \mathcal{X}' \rightarrow \mathcal{X} \rightarrow \mathcal{X}'' \rightarrow 0$$

holds

$$\max\{\mathfrak{c}(\mathcal{X}'), \mathfrak{c}(\mathcal{X}'')\} \leq \mathfrak{c}(\mathcal{X}) \leq \mathfrak{c}(\mathcal{X}') + \mathfrak{c}(\mathcal{X}'') \quad (3-21)$$

and

$$\mathfrak{c}(\mathcal{X}) = \max\{\mathfrak{c}(\mathcal{X}'), \mathfrak{c}(\mathcal{X}'')\} \quad (3-22)$$

if the exact sequence above splits. In particular,

$$\mathfrak{c}(\mathcal{X} \oplus \mathcal{X}) = \mathfrak{c}(\mathcal{X}) \quad (3-23)$$

for any torsion \mathcal{X} .

Cf. [Fa1], Proposition 4.9. The arguments of [Fa1] are in fact based on results of [LL].

3.11. Corollary. *For a fixed trace tr on a finite von Neumann category, and for a given $\nu \in [0, \infty]$, let $\mathcal{T}_\nu(\mathcal{C}_\mathcal{A})$ denote the full subcategory of $\mathcal{T}(\mathcal{C}_\mathcal{A})$ whose objects are all torsion objects \mathcal{X} of $\mathcal{E}(\mathcal{C}_\mathcal{A})$ with capacity (taken with respect to the trace tr) $\mathfrak{c}(\mathcal{X}) \leq \nu$. Then $\mathcal{T}_\nu(\mathcal{C}_\mathcal{A})$ is an abelian subcategory. \square*

3.12. Example. *In a finite von Neumann category there may exist non-isomorphic torsion objects which have identical Novikov-Shubin numbers with respect to any trace on the initial von Neumann category.*

To show this, consider the von Neumann category $\mathcal{C}_\mathcal{A}$ of measurable fields of finite dimensional Hilbert spaces over the circle S^1 supplied with the Lebesgue measure. The algebra $\mathcal{A} = L^\infty(S^1)$ clearly acts on the objects of this category $\mathcal{C}_\mathcal{A}$. The traces on this category are in one-to-one correspondence with the measures on the circle S^1 , which are absolutely continuous with respect to the Lebesgue measure.

Let $\mathcal{H} \in \text{ob}(\mathcal{C}_\mathcal{A})$ denote the object represented by a constant field of one-dimensional spaces over the circle. The corresponding Hilbert space of L^2 sections is $\mathcal{H} = L^2(S^1)$. For any angle $\theta \in [0, 2\pi]$ and for any real $\nu > 0$ consider the following torsion object $\mathcal{X} = \mathcal{X}_{\theta, \nu} = (\alpha : \mathcal{H} \rightarrow \mathcal{H})$, where α is the operator of pointwise multiplication by the function $\alpha(z) = |z - e^{i\theta}|^\nu$, where $z \in S^1$.

It follows from the claim in section 7.4 of [Fa1] that \mathcal{X} and $\mathcal{X} \oplus \mathcal{X}$ are not isomorphic. But they have identical capacities with respect to any trace according to Proposition 3.10 above.

3.13. Duality for torsion objects. Let $\mathcal{C}_\mathcal{A}$ be a finite von Neumann category, and let $\mathcal{X} = (\alpha : A' \rightarrow A)$ be a torsion object, such that α is injective. We will define the *dual object* $\mathfrak{e}(\mathcal{X})$ as $\mathfrak{e}(\mathcal{X}) = (\alpha^* : A \rightarrow A')$. Here the star denotes the adjoint morphism. If $\mathcal{Y} = (\beta : B' \rightarrow B)$ is another torsion object with injective β then any morphism $[f] : \mathcal{X} \rightarrow \mathcal{Y}$ represented by a morphism $f : A \rightarrow B$ in $\mathcal{C}_\mathcal{A}$, determines the dual morphism $\mathfrak{e}([f]) : \mathfrak{e}(\mathcal{Y}) \rightarrow \mathfrak{e}(\mathcal{X})$ which is defined as follows. According to the definition of section 1.3, there exists a morphism $g : A' \rightarrow B'$ in $\mathcal{C}_\mathcal{A}$ such that $f \circ \alpha = \beta \circ g$; this g is in fact unique since α and β are supposed to be injective. Then $\mathfrak{e}([f])$ is defined as morphism $[g^*]$.

The duality \mathfrak{e} is a contravariant functor $\mathfrak{e} : \mathcal{T}(\mathcal{C}_\mathcal{A}) \rightarrow \mathcal{T}(\mathcal{C}_\mathcal{A})$, and there is natural equivalence $\mathfrak{e} \circ \mathfrak{e} \simeq \text{Id}$.

We refer to [Fa1], section 3.8 for a more detailed description of this duality in a specific situation.

Observe that *any torsion object is isomorphic to its dual*, but this isomorphism is not canonical. In fact, by Proposition 2.3, we know that any torsion object is isomorphic to $(\alpha : A \rightarrow A)$; then using the polar decomposition we may replace α by a self-adjoint map $\alpha^* = \alpha$. In this case the dual of $\mathcal{X} = (\alpha : A \rightarrow A)$ is identical to \mathcal{X} .

3.14. Computing the Novikov - Shubin invariants of cohomology. In many geometric applications torsion objects appear as cohomology of projective cochain complexes. We need to be able to compute their spectral density functions and the Novikov - Shubin numbers in terms of the given chain complex. We are going to make some general remarks concerning this question, which will be used later in applications.

Suppose that $\mathcal{C}_\mathcal{A}$ is a finite von Neumann category and tr is a fixed trace in $\mathcal{C}_\mathcal{A}$.

Let

$$C^* = (\dots \rightarrow C^{i-1} \xrightarrow{\partial} C^i \xrightarrow{\partial} C^{i+1} \xrightarrow{\partial} \dots) \quad (3-24)$$

be a cochain complex in $\mathcal{C}_{\mathcal{A}}$. By Proposition 1.10 we have the following formula for the extended L^2 cohomology

$$\mathcal{H}^i(C) = (\partial : C^{i-1} \rightarrow Z^i) \quad (3-25)$$

where $Z^i = \ker[\partial : C^i \rightarrow C^{i+1}]$. Applying Definition 3.7, we obtain the following recipe of computing the spectral density function of the torsion part of $\mathcal{H}^i(C)$. Consider the self-adjoint operator

$$\partial^* \partial : C^{i-1} \rightarrow C^{i-1} \quad (3-26)$$

and let

$$G(\lambda) = \text{tr}(\chi_{[0, \lambda]}(\partial^* \partial)), \quad (3-27)$$

where $\chi_{[0, \lambda]}$ denotes the characteristic function of the interval $[0, \lambda]$. Then *the von Neumann dimension of the projective part of $\mathcal{H}^i(C)$ equals*

$$\dim_{\text{tr}} P(\mathcal{H}^i(C)) = G(0) \quad (3-28)$$

and the spectral density function of the torsion part of $\mathcal{H}^i(C)$ equals

$$F^i(\lambda) = G(\lambda^2) - G(0). \quad (3-29)$$

We shall also point out the relation between the spectral density function of the Laplacian $\Delta^i = \partial \partial^* + \partial^* \partial : C^i \rightarrow C^i$ and the functions $F^i(\lambda)$ which appear above. Namely,

$$\text{tr}(\chi_{[0, \lambda]}(\Delta^i)) = F^i(\sqrt{\lambda}) + F^{i+1}(\sqrt{\lambda}). \quad (3-30)$$

We will leave the proof of this formula as an easy exercise.

§4. Divisor determined by a torsion object

This section devoted to studying "*families*" or "*the abelian case*". Namely, let Z be a locally compact Hausdorff space and let μ be a positive Radon measure on Z . In this section we will deal with a von Neumann category $\mathcal{C}_{\mathcal{A}}$ of Hilbert representations of the algebra $\mathcal{A} = L_{\mathbb{C}}^{\infty}(Z, \mu)$ of essentially bounded complex valued functions on Z . We will see that any torsion object \mathcal{X} of the corresponding extended category $\mathcal{E}(\mathcal{C}_{\mathcal{A}})$ determines a "divisor" - a closed subset $\mathcal{D}(\mathcal{X}) \subset Z$ with additional information on it ("*multiplicities*"). We compute explicitly the divisor assuming that the torsion object is given as the cohomology of a chain complex of vector bundles.

We compute also some simple examples which provide an evidence that by considering torsion objects of the extended category it is possible to capture all relevant information about intersections of analytic varieties.

4.1. Let $\mathcal{C}_{\mathcal{A}}$ denote the following von Neumann category of Hilbert representations of $*$ -algebra $\mathcal{A} = L_{\mathbb{C}}^{\infty}(Z, \mu)$. The objects of category $\mathcal{C}_{\mathcal{A}}$ are in one-to-one correspondence with measurable fields of finite dimensional Hilbert spaces $\xi \rightarrow \mathcal{H}(\xi)$ over Z such that the dimension $\dim \mathcal{H}(\xi)$ is essentially bounded. Any such field $\mathcal{H}(\xi)$ determines a Hilbert representation of \mathcal{A} acting on the direct integral of Hilbert spaces (2-4). The morphisms of $\mathcal{C}_{\mathcal{A}}$ are bounded linear maps between the corresponding Hilbert spaces given by decomposable linear maps (2-5), cf. example 7, section 2.6 (this category was denoted $\mathcal{C}'_{\mathcal{A}}$ then).

From section 2.7 we know that any measure μ' on Z which is absolutely continuous with respect to μ and such that $\mu'(Z) < \infty$ determines a trace $\text{tr}_{\mu'}$ on von Neumann category $\mathcal{C}_{\mathcal{A}}$, cf. (2-11).

Let $\mathcal{E}(\mathcal{C}_{\mathcal{A}})$ denote the extended category of $\mathcal{C}_{\mathcal{A}}$ and let \mathcal{X} be a torsion object of $\mathcal{E}(\mathcal{C}_{\mathcal{A}})$. Applying construction of section 3.7, we obtain that any measure μ' as in the previous paragraph determines a trace $\text{tr}_{\mu'}$ on category $\mathcal{C}_{\mathcal{A}}$ and, the corresponding spectral density function of \mathcal{X} with respect to $\text{tr}_{\mu'}$; we denote it by $F_{\mu'}(\lambda)$.

4.2. Given a torsion object \mathcal{X} of $\mathcal{E}(\mathcal{C}_{\mathcal{A}})$ we define *its divisor* $\mathcal{D}(\mathcal{X}) \subset Z$ as the set of all points $\xi \in Z$ such that for any neighbourhood V of ξ there exists a measure $\mu' \ll \mu$ with support of μ' contained in V and such that the corresponding spectral density function $F_{\mu'}(\lambda)$ is *strictly positive* for all $\lambda > 0$.

$\mathcal{D}(\mathcal{X})$ is a closed set. To see this, one just forms the negation of the definition in the previous paragraph: $\xi \notin \mathcal{D}(\mathcal{X})$ iff there exists a neighbourhood V of ξ such that for any measure $\mu' \ll \mu$ with support in V the corresponding spectral density function $F_{\mu'}(\lambda)$ of \mathcal{X} vanishes for all small $\lambda > 0$.

Proposition 3.8 implies that *any two isomorphic torsion objects \mathcal{X} and \mathcal{Y} have identical divisors $\mathcal{D}(\mathcal{X}) = \mathcal{D}(\mathcal{Y})$.*

The following statement computes the divisor in many interesting situations.

4.3. Proposition. *Let Z be a compact Hausdorff space supplied with a positive Radon measure μ such that there are no nonempty open sets $U \subset Z$ with $\mu(U) = 0$. Let \mathcal{E} be a Hermitian vector bundle over Z of rank n (i.e. a locally trivial vector bundle such that each fiber is given a Hermitian metric which varies continuously). Let $T : \mathcal{E} \rightarrow \mathcal{E}$ be a continuous map which induces a linear self-map on each fiber \mathcal{E}_{ξ} , where $\xi \in Z$. For any point $\xi \in Z$ denote by $d(\xi)$ the determinant of the linear map $T(\xi) : \mathcal{E}_{\xi} \rightarrow \mathcal{E}_{\xi}$ of the fiber above ξ . Suppose that the zero set $\{\xi \in Z; d(\xi) = 0\}$ of the obtained continuous function $d : Z \rightarrow \mathbb{C}$ has measure zero. Then the induced by T map T^* of the spaces of L^2 sections $\mathcal{X} = (T^* : L^2(\mathcal{E}) \rightarrow L^2(\mathcal{E}))$ represents a torsion object of the extended category $\mathcal{E}(\mathcal{C}_{\mathcal{A}})$ and the divisor $\mathcal{D}(\mathcal{X})$ of \mathcal{X} coincides with the zero set $\{\xi \in Z; d(\xi) = 0\}$ of the function d .*

Proof. The linear map $T(\xi)$ is invertible $S(\xi) = T^{-1}(\xi)$ for $d(\xi) \neq 0$. We can find a sequence of open sets $U_1 \supset U_2 \supset U_3 \supset \dots$ with $\mu(U_j) \rightarrow 0$, which contain the zero set $\{\xi \in Z; d(\xi) = 0\}$ and such that all the spaces $L^2(\mathcal{E}|_{Z-U_n})$ belong to the image of T^* and their union $\cup L^2(\mathcal{E}|_{Z-U_n})$ is dense in $L^2(\mathcal{E})$. This shows that the map T^* has dense image and so \mathcal{X} is a torsion object.

Suppose that $\xi \in Z$ is such that $d(\xi) \neq 0$. Then there is a neighborhood U of ξ such that the bundle map $T|_U : \mathcal{E}|_U \rightarrow \mathcal{E}|_U$ is invertible. Then clearly, for any measure $\mu' \ll \mu$ with $\text{supp}(\mu') \subset U$ we will have $F_{\mu'}(\lambda) = 0$ for small $\lambda > 0$. Thus $U \cap \mathcal{D}(\mathcal{X}) = \emptyset$. This proves that the divisor $\mathcal{D}(\mathcal{X})$ is contained in the zero set of d .

Suppose now that $\xi \in Z$ and $d(\xi) = 0$. We want to show that for any closed neighborhood U of ξ the spectral density function $F_{\mu'}(\lambda)$ is strictly positive for all $\lambda > 0$, where $\mu' = \mu|_U$. Suppose the contrary, i.e. there exists a closed neighborhood U of ξ and a positive number λ_U such that taking $\mu' = \mu|_U$ we have for the corresponding spectral density function $F_{\mu'}(\lambda) = 0$ for all $0 < \lambda < \lambda_U$. Restricting the neighborhood U if necessary, we may additionally assume that

$$|d(\xi)| < \lambda_U^n \quad (4-1)$$

for all $\xi \in U$. Also, we may assume that the bundle \mathcal{E} can be trivialized over U and so $T(\xi)$ is given by an $n \times n$ -matrix $(a_{ij}(\xi))$ of continuous functions of $\xi \in U$. Consider the non-negative self-adjoint matrix

$$(T^*T)(\xi) = (c_{ij}(\xi)), \quad \text{where} \quad c_{ij}(\xi) = \sum_{k=1}^n \bar{a}_{ik}(\xi) a_{kj}(\xi). \quad (4-2)$$

We may find measurable functions $\lambda_i : U \rightarrow \mathbb{R}$, where $i = 1, 2, \dots, n$, such that the numbers

$$0 \leq \lambda_1(\xi) \leq \lambda_2(\xi) \leq \dots \leq \lambda_n(\xi) \quad (4-3)$$

form the set of eigenvalues of the matrix (4-2) for almost all $\xi \in U$. Applying the definition of spectral density function in section 3.7, we obtain

$$F_{\mu'}(\lambda) = \sum_{i=1}^n \mu\{\xi \in U; \lambda_i(\xi) \leq \lambda^2\}. \quad (4-4)$$

Now, our assumption that $F_{\mu'}(\lambda) = 0$ for all $\lambda < \lambda_U$ implies that for any $i = 1, 2, \dots, n$ holds

$$\mu\{\xi \in U; \lambda_i(\xi) > \lambda_U^2\} = \mu(U), \quad (4-5)$$

and so

$$\mu\left(\bigcap_{i=1}^n \{\xi \in U; \lambda_i(\xi) > \lambda_U^2\}\right) = \mu(U). \quad (4-6)$$

However

$$d(\xi)^2 = \prod_{i=1}^n \lambda_i(\xi) < \lambda_U^{2n} \quad (4-7)$$

almost everywhere on U , which gives a contradiction. \square

4.4. Example. Consider the special case of the situation of the previous Proposition when \mathcal{E} is the trivial line bundle. Then the bundle map T is given by a continuous function $f : Z \rightarrow \mathbb{C}$ and the corresponding operator $T^* : L^2(Z) \rightarrow L^2(Z)$ is multiplication by f , which we will denote M_f . Proposition 4.3 states that in this case the divisor of the torsion object $\mathcal{X} = \mathcal{X}_f = (M_f : L^2(Z) \rightarrow L^2(Z))$ equals to the zero set $\{\xi \in Z; f(\xi) = 0\}$, provided the measure of the zero set with respect to μ vanishes.

Let us take, for instance, $Z = S^1$, and the measure μ being the Lebesgue measure. Let $f : S^1 \rightarrow \mathbb{C}$ be the following function $f(z) = |z - e^{i\theta}|^\nu$, where $\theta \in [0, 2\pi]$ and $\nu > 0$. Denote the corresponding torsion object $\mathcal{X}_{\theta, \nu}$ (it appeared already in section 3.12.) The divisor $\mathcal{D}(\mathcal{X}_{\theta, \nu})$ is the one-point set $\{\theta\}$. This shows that for $\theta \neq \theta'$ the objects $\mathcal{X}_{\theta, \nu}$ and $\mathcal{X}_{\theta', \nu}$ are not-isomorphic, cf. also [Fa1].

Note also that the Novikov - Shubin number $\mathbf{ns}(\mathcal{X}_{\theta, \nu})$ with respect to the Lebesgue measure on S^1 equals ν^{-1} , cf. [Fa1]. Thus, one may find the number ν by means of the Novikov - Shubin number.

4.5. Corollary. *In the situation of Proposition 4.3, the condition $\mathcal{D}(\mathcal{X}) = \emptyset$ implies $\mathcal{X} = 0$.*

Indeed, $\mathcal{D}(\mathcal{X}) = 0$ implies, by Proposition 4.3, that the determinant function d nowhere vanishes and thus $T(\xi)$ is invertible for all ξ . Because Z is compact we obtain that the induced map on the spaces of L^2 -sections $T^* : L^2(\mathcal{E}) \rightarrow L^2(\mathcal{E})$ is invertible and so $\mathcal{X} = 0$.

Note that this corollary is not true if Z is only locally compact. Consider the following example. Let $Z = \mathbb{R}$ with the measure $\mu = dx/x^2$, and let $f(x) = e^{-x^2}$. Since f is bounded, it induces a map between the spaces of L^2 -section. Then the torsion object $\mathcal{X}_f = (M_f : L^2(Z, \mu) \rightarrow L^2(Z, \mu))$ has empty divisor $\mathcal{D}(\mathcal{X}) = \emptyset$, but $\mathcal{X} \neq 0$.

This suggests the idea to study "the divisor at infinity" as well.

4.6. Proposition. *(Integrality of the Novikov - Shubin capacity) Let Z be a compact smooth manifold and let $f : Z \rightarrow \mathbb{R}$ be a smooth function which is transversal with respect to $0 \in \mathbb{R}$. For any positive integer $m > 0$ consider the following torsion object*

$$\mathcal{X}_{f^m} = (M_{f^m} : L^2(Z) \rightarrow L^2(Z)),$$

where the $L^2(Z)$ is constructed with respect to a smooth measure on Z . Then the Novikov - Shubin capacity $\mathfrak{c}(\mathcal{X}_{f^m})$ with respect to any smooth measure μ on Z is integral and equals m . Moreover, the spectral density function of \mathcal{X}_{f^m} is dilatationally equivalent to $\lambda^{1/m}$.

Proof. Let $\mathcal{D} = \mathcal{D}(\mathcal{X}_{f^m}) = \{\xi \in Z; f(\xi) = 0\}$ denote the divisor of \mathcal{X} . It is a smooth codimension one submanifold of Z . The spectral density function $F_\mu(\lambda)$ of \mathcal{X}_{f^m} is

$$F_\mu(\lambda) = \mu(U_\lambda), \quad \text{where } U_\lambda = \{\xi \in Z; |f(\xi)|^m \leq \lambda\}. \quad (4-8)$$

From this formula it clear that it is enough to prove Proposition 4.6 in case $m = 1$; therefore we will assume that $m = 1$ in the rest of the proof.

Fix a Riemannian metric on Z . We will denote by μ the corresponding Riemannian measure on Z . Because of the transversality assumption and compactness of the divisor \mathcal{D} , we may find positive constants c and C such that

$$c < |df(\xi)| < C \quad (4-9)$$

holds for all ξ in some open neighborhood W of \mathcal{D} . Consider a smooth vector field X on Z which is orthogonal to \mathcal{D} on \mathcal{D} and such that the vectors of the field $X(\xi)$ have unit length for $\xi \in W$. Since $\langle df, X \rangle = \pm |df|$ on \mathcal{D} , we may assume that the neighborhood W is so small that

$$c/2 < |\langle df(\xi), X(\xi) \rangle| < C \quad (4-10)$$

holds for all points in W .

The neighborhood W is fibred by integral trajectories of the field X starting from the points of \mathcal{D} . If a point $x \in W$ belongs to a trajectory $x(t)$ starting at a point $p \in \mathcal{D}$ then using $\frac{d}{dt} f(x(t)) = \langle df, X \rangle$ and (4-10) we obtain

$$c\rho(x)/2 \leq |f(x)| \leq C\rho(x), \quad (4-11)$$

where $\rho(x)$ denotes the Riemannian distance between x and p along the trajectory.

Thus we obtain that for small λ the set U_λ contains $\{\xi \in W; \rho(\xi) \leq \lambda C^{-1}\}$; similarly, for small λ the set U_λ is contained in $\{\xi \in W; \rho(\xi) \leq 2\lambda c^{-1}\}$. Therefore

$$\lambda AC^{-1} \leq F_\mu(\lambda) \leq 2\lambda Ac^{-1}, \quad (4-12)$$

where A denotes the area of \mathcal{D} .

(4-12) shows that $F_\mu(\lambda) \sim \lambda$. \square

When the divisor is not a submanifold the capacity may be not integral and the spectral density function may be not equivalent to a power λ^α . We consider a few illustrating examples.

4.7. Example: Transversal crossing. Consider the 2-dimensional torus $T = \mathbb{R}/\mathbb{Z} \times \mathbb{R}/\mathbb{Z}$ with coordinates x and y , defined modulo integers. Let $f : T \rightarrow T$ be the function $f(x, y) = xy$. Consider the following torsion object $\mathcal{X} = (M_f : L^2(T) \rightarrow L^2(T))$. Then the divisor $\mathcal{D}(\mathcal{X})$, which is equal to $\{(x, y); x = 0 \text{ or } y = 0\}$, has a singular point $x = 0, y = 0$ (the cross). An easy computation of the spectral density function of \mathcal{X} with respect to the Lebesgue measure μ gives $F_\mu(\lambda) \sim \lambda(1 - \log \lambda)$ which has capacity 1, but *it is not dilatationally equivalent to $\sim \lambda$* .

4.8. Example: Tangency. Let $Z = [-1, 1] \times [-1, 1]$ be the two-dimensional square with the coordinates x and y supplied with the Lebesgue measure. Let k be a non-negative integer. Consider the function $f : Z \rightarrow \mathbb{R}$ given by $f(x, y) = y(y - x^k)$. As above, it defines the following torsion object $\mathcal{X} = (M_f : L^2(Z) \rightarrow L^2(Z))$. The divisor $\mathcal{D}(\mathcal{X})$ is the union of the interval $[-1, 1]$ of the real axis and the parabola $y = x^k$; the divisor has a singular point at the origin. A computation of the capacity $\mathfrak{c}(\mathcal{X})$ in this example gives

$$\mathfrak{c}(\mathcal{X}) = \frac{2k}{k+1}. \quad (4-13)$$

This shows that the order of tangency k can be recovered from the capacity $\mathfrak{c}(\mathcal{X})$. Note also that in this example $1 < \mathfrak{c}(\mathcal{X}) < 2$ for $k > 1$ and $\mathfrak{c}(\mathcal{X}) \rightarrow 2$ when $k \rightarrow \infty$.

4.9. Example: Divisors of higher codimension. If the divisor $\mathcal{D}(\mathcal{X})$ is a submanifold but of codimension greater than 1, then the capacity $\mathfrak{c}(\mathcal{X})$ may be rational and not integral.

Consider, for example the n -dimensional torus T^n with coordinates x_1, x_2, \dots, x_n considered modulo \mathbb{Z} and the function $f(x) = (\sum_{i=1}^n x_i^2)^m$, where $m > 0$ is an integer. Then the divisor of the corresponding torsion object \mathcal{X}_f consists of one point 0, and the Novikov - Shubin capacity $\mathfrak{c}(\mathcal{X}_f)$ equals $2m/n$. Moreover, the spectral density function is the power $\sim \lambda^{n/2m}$.

Now we formulate the following generalization of Proposition 4.3.

4.10. Theorem. *Let Z be a compact Hausdorff space supplied with a positive Radon measure μ such that there are no nonempty open sets $U \subset Z$ with $\mu(U) = 0$. Let \mathcal{E}^i , where $i = 0, 1, 2, \dots, N$, be a finite sequence of Hermitian vector bundles over Z . Suppose that for each i there is given a continuous bundle map $\partial^i : \mathcal{E}^i \rightarrow \mathcal{E}^{i+1}$ such that the sequence*

$$(\mathcal{E}, \partial) = (0 \rightarrow \mathcal{E}^0 \xrightarrow{\partial^0} \mathcal{E}^1 \xrightarrow{\partial^1} \mathcal{E}^2 \dots \xrightarrow{\partial^{(N-1)}} \mathcal{E}^N \rightarrow 0) \quad (4-14)$$

represents a cochain complex, i.e. $\partial^{i+1} \circ \partial^i = 0$. Denote by $\mathcal{N}(\mathcal{E}, \partial)$ the set of all points $\xi \in Z$, such that the complex $(\mathcal{E}_\xi, \partial_\xi)$ of the fibers over ξ is not acyclic. Suppose that $\mathcal{N}(\mathcal{E}, \partial)$ has μ -measure zero. Then the extended cohomology $\mathcal{H}^*(L^2(\mathcal{E}), \partial^*)$ of the induced complex of L^2 -sections

$$(L^2(\mathcal{E}), \partial^*) = (0 \rightarrow L^2(\mathcal{E}^0) \xrightarrow{\partial^{0*}} L^2(\mathcal{E}^1) \xrightarrow{\partial^{1*}} \dots \xrightarrow{\partial^{(N-1)*}} L^2(\mathcal{E}^N) \rightarrow 0) \quad (4-15)$$

(understood as a graded object of the extended category $\mathcal{E}(\mathcal{C}_A)$) is torsion in all dimensions and the union of the divisors

$$\bigcup_{i=0}^N \mathcal{D}(\mathcal{H}^i(L^2(\mathcal{E}), \partial^*)) \quad (4-16)$$

coincides with $\mathcal{N}(\mathcal{E}, \partial)$. Moreover, the extended cohomology $\mathcal{H}^*(L^2(\mathcal{E}), \partial^*)$ vanishes if and only if the set $\mathcal{N}(\mathcal{E}, \partial)$ is empty.

Proof. We will deduce this statement from Proposition 4.3.

First we observe that the set $\mathcal{N}(\mathcal{E}, \partial)$ is obviously closed. If $\xi \notin \mathcal{N}(\mathcal{E}, \partial)$ then we can find a compact neighborhood U of ξ such that $U \cap \mathcal{N}(\mathcal{E}, \partial) = \emptyset$ and then the complex of bundles $(\mathcal{E}|_U, \partial|_U)$ is acyclic. Moreover, using induction we can construct step by step a null-homotopy $S^i : \mathcal{E}^i|_U \rightarrow \mathcal{E}^{i-1}|_U$ by means of continuous bundle maps S^i such that $S\partial + \partial S = \text{id}$. Therefore, the corresponding complex of L^2 -sections, constructed out of the complex $(\mathcal{E}|_U, \partial|_U)$, is acyclic if considered as complex in the extended category. Thus, we obtain that for any measure $\mu' \ll \mu$ with $\text{supp}(\mu') \subset U$ the spectral density functions of extended cohomology of complex (4-15) vanish. Since the complement $Z - \mathcal{N}(\mathcal{E}, \partial)$ is of full measure, this shows that the extended cohomology of (4-15) is torsion and the union of the divisors (4-16) is contained in $\mathcal{N}(\mathcal{E}, \partial)$.

Suppose now that $\xi \in \mathcal{N}(\mathcal{E}, \partial)$. Then the complex of fibers $(\mathcal{E}_\xi, \partial_\xi)$ over ξ has non-trivial cohomology in some dimension i . Consider the i -dimensional Laplacian: $\Delta^i : \mathcal{E}^i \rightarrow \mathcal{E}^i$. Here Δ^i denotes $\hat{\partial}^i \circ \partial^i + \partial^{i-1} \circ \hat{\partial}^{i-1}$, where the sign "hat" denotes the adjoint bundle map with respect to the given Hermitian structures. Δ^i is a continuous bundle map $\mathcal{E}^i \rightarrow \mathcal{E}^i$, and so we may apply Proposition 4.3 to it. Since at point ξ the complex of the fibers $(\mathcal{E}_\xi, \partial_\xi)$ has nontrivial cohomology in dimension i , it follows that the i -dimensional Laplacian at the point ξ (which equals to evaluation of Δ^i at ξ) has a non-trivial kernel. By Proposition 4.3 we obtain that then for any neighborhood U of ξ one may find a measure $\mu' \ll \mu$ with $\text{supp}(\mu') \subset U$ such that the spectral density function of the torsion object

$$(\Delta^{i*} : L^2(\mathcal{E}^i) \rightarrow L^2(\mathcal{E}^i)) \quad (4-17)$$

is strictly positive for positive λ . But then formula (3-30) implies that ξ belongs to the divisor of the torsion object $\mathcal{T}(\mathcal{H}^j(L^2(\mathcal{E}), \partial^*))$ for j equal i or $i+1$. \square

In the analytic situation we can strengthen Theorem 4.10 by abandoning the assumption that the complex of bundles (\mathcal{E}, ∂) is acyclic almost everywhere:

4.11. Theorem. *Let Z be a compact real analytic manifold, and let*

$$(\mathcal{E}, \partial) = (0 \rightarrow \mathcal{E}^0 \xrightarrow{\partial^0} \mathcal{E}^1 \xrightarrow{\partial^1} \mathcal{E}^2 \dots \xrightarrow{\partial^{(N-1)}} \mathcal{E}^N \rightarrow 0) \quad (4-18)$$

be a cochain complex of real analytic Hermitian vector bundles over Z and real analytic bundle maps between them. For any point $\xi \in Z$ consider the complex $(\mathcal{E}_\xi, \partial_\xi)$ of the fibers above ξ and denote by $\beta^i(\xi)$ the i -dimensional Betti number of this chain complex. Then there is a minimal subset $\mathcal{N}(\mathcal{E}, \partial) \subset Z$ of measure zero such that all the functions $\beta^i(\xi)$, $i = 0, 1, \dots, N$ are constant for $\xi \in Z - \mathcal{N}(\mathcal{E}, \partial)$. We will denote by β^i the common value of $\beta^i(\xi)$ for $\xi \in Z - \mathcal{N}(\mathcal{E}, \partial)$; we will refer to it as to generic Betti number.

Fix a smooth measure μ on Z coming from a Riemannian metric, and consider the complex of L^2 sections $(L^2(\mathcal{E}), \partial^)$:*

$$(0 \rightarrow L^2(\mathcal{E}^0, \mu) \xrightarrow{\partial^{0*}} L^2(\mathcal{E}^1, \mu) \xrightarrow{\partial^{1*}} \dots \xrightarrow{\partial^{(N-1)*}} L^2(\mathcal{E}^N, \mu) \rightarrow 0) \quad (4-19)$$

which can be viewed as a projective chain complex in \mathcal{C}_A . Then the extended cohomology $\mathcal{H}^(L^2(\mathcal{E}), \partial^*)$ of this complex has the following properties:*

- (1) *the von Neumann dimension of the projective part of the extended cohomology of complex (4-19) is given by*

$$\dim_{\text{tr}} P(\mathcal{H}^i(L^2(\mathcal{E}), \partial^*)) = \mu(Z)\beta^i, \quad (4-20)$$

where tr denotes the trace on the von Neumann category \mathcal{C}_A constructed by means of measure μ (cf. 2.7);

- (2) *the union of the divisors of the torsion part of the extended cohomology is given by*

$$\bigcup_{i=0}^N \mathcal{D}(\mathcal{T}(\mathcal{H}^i(L^2(\mathcal{E}), \partial^*))) = \mathcal{N}(\mathcal{E}, \partial); \quad (4-21)$$

- (3) *the extended cohomology of the L^2 complex (4-19) vanishes if and only if the complex $(\mathcal{E}_\xi, \partial_\xi)$ is acyclic for all $\xi \in Z$.*

Proof. Consider first a special case of Theorem 4.11, when it can be trivially deduced from Theorem 4.10.

Suppose that the given chain complex (\mathcal{E}, ∂) over Z can be represented as a direct sum $(\mathcal{E}_0, \partial) \oplus (\mathcal{E}_+, \partial)$, where $(\mathcal{E}_0, \partial)$ and $(\mathcal{E}_+, \partial)$ are two chain complexes over Z formed by vector bundles and bundle maps, such that the first complex $(\mathcal{E}_0, \partial)$ is acyclic almost everywhere on Z (as in Theorem 4.10) and the second complex of vector bundles $(\mathcal{E}_+, \partial)$ has trivial differential. In this case we will say that the original chain complex (\mathcal{E}, ∂) *splits*.

It is clear that in the split case Theorem 4.11 is true. In fact, for any i the rank of \mathcal{E}_+^i equals to the generic Betti number β^i ; also the complex of L^2 sections (4-19) is the direct sum $(L^2(\mathcal{E}_0), \partial^*) \oplus (L^2(\mathcal{E}_+), \partial^*)$ and so the extended cohomology of $(L^2(\mathcal{E}), \partial^*)$ is the direct sum of the extended cohomology of $(L^2(\mathcal{E}_0), \partial^*)$ and $(L^2(\mathcal{E}_+), \partial^*)$. We may apply Theorem 4.10 to compute the extended cohomology of $(L^2(\mathcal{E}_0), \partial^*)$. The

extended cohomology of the second complex $(L^2(\mathcal{E}_+, \partial)$ in each dimension j coincides with $L^2(\mathcal{E}_+^j)$.

Let us show that one may reduce the general case to the split case by performing certain *blow up*.

Our strategy will be as follows. Given a complex (\mathcal{E}, ∂) of real analytic vector bundles as above, we will construct a compact Hausdorff space \hat{Z} and a continuous map $\phi : \hat{Z} \rightarrow Z$ such that

- (1) the induced complex of vector bundles $(\phi^*(\mathcal{E}), \partial)$ over \hat{Z} splits (in the sense explained above);
- (2) ϕ is onto and for any point $p \in Z - \mathcal{N}(\mathcal{E}, \partial)$ the preimage $\phi^{-1}(p)$ consists of precisely one point;
- (3) the set $\mathcal{N}(\phi^*(\mathcal{E}), \partial) \subset \hat{Z}$ of *the special values* for the induced chain complex coincides with $\phi^{-1}(\mathcal{N}(\mathcal{E}, \partial))$;
- (4) the interior of the set $\phi^{-1}(\mathcal{N}(\mathcal{E}, \partial))$ is empty and there exists a positive Radon measure $\hat{\mu}$ on \hat{Z} such that for any measurable subset $U \subset Z'$ we have $\mu'(U) = \mu(\phi(U))$.

Since $\mathcal{N}(\mathcal{E}, \partial)$ has measure zero (as a proper real analytic subset) we obtain that the canonical map $L^2(\mathcal{E}^j, \mu) \rightarrow L^2(\phi^*(\mathcal{E}^j), \mu')$ is an isomorphism of Hilbert $L^\infty(Z, \mu)$ -modules and so we may identify the extended cohomology of complex (4-19) with the extended cohomology of complex $(L^2(\phi^*(\mathcal{E}), \partial)$. The last complex splits, and so its extended cohomology is given by the remark above.

We will now show how one can construct a blow up $\phi : \hat{Z} \rightarrow Z$ with the above properties.

For each i let $G_{\beta^i}(\mathcal{E}^i) \rightarrow Z$ denote the bundle of β^i -dimensional Grassmannians associated to the bundle \mathcal{E}^i . Let β denote the vector of generic Betti numbers $\beta = (\beta^0, \beta^1, \dots, \beta^N)$ and let $G_\beta(\mathcal{E}) \rightarrow Z$ denote the fiber product

$$G_\beta(\mathcal{E}) = G_{\beta^0}(\mathcal{E}^0) \times_Z G_{\beta^1}(\mathcal{E}^1) \times_Z \cdots \times_Z G_{\beta^N}(\mathcal{E}^N). \quad (4-22)$$

A point $L \in G_\beta(\mathcal{E})$ above some point $\xi \in Z$ is represented by a sequence $L = (L^0, L^1, \dots, L^N)$, where L^i is a β^i -dimensional subspace in the fiber \mathcal{E}_ξ^i above ξ . We will denote by π the projection $G_\beta(\mathcal{E}) \rightarrow Z$. Thus, $\xi = \pi(L)$ in the notations above.

Denote by Z' the subset of $G_\beta(\mathcal{E})$ consisting of all points $L = (L^0, L^1, \dots, L^N) \in G_\beta(\mathcal{E})$ such that:

- (a) $\partial_\xi(L^i) = 0$ for all $i = 0, 1, \dots, N-1$, where $\xi = \pi(L)$;
- (b) $\delta_\xi(L^i) = 0$ for all $i = 1, 2, \dots, N$.

Here δ_ξ denotes the adjoint of ∂_ξ with respect to the given Hermitian metrics on the fibers above ξ . The condition (a) means that the subspace L^i is contained in $\ker(\partial_\xi)$ and similarly for (b).

Denote by $\phi : Z' \rightarrow Z$ the restriction of π .

We see that Z' is compact as a closed subset of the Grassmannian bundle $G_\beta(\mathcal{E})$.

If ξ is a generic point of Z (i.e. $\xi \notin \mathcal{N}(\mathcal{E}, \partial)$) then the preimage $\phi^{-1}(\xi)$ consists of precisely one point $L = (L^0, L^1, \dots, L^N)$, where L^i is the space of harmonic vectors in the fiber \mathcal{E}_ξ^i .

Now we may finally define the blow up \hat{Z} as the closure of $\phi^{-1}(Z - \mathcal{N}(\mathcal{E}, \partial))$. We will also denote by $\phi|_{\hat{Z}}$ the restriction map $\phi|_{\hat{Z}}$.

The induced chain complex of bundles $(\phi^*(\mathcal{E}), \partial)$ over \hat{Z} splits; its splitting over a point $L \in \hat{Z}$ is given by assigning $(\phi^*(\mathcal{E}^i)_+)_L = L^i$ and $(\phi^*(\mathcal{E}^i)_0)_L = L^{i\perp}$.

It is easy to see that the conditions (1) - (4) mentioned above are satisfied. \square

Note, that as follows from the construction, the bow up \hat{Z} has structure of a real algebraic set.

4.12. Novikov - Shubin invariants and the germ-cohomology. Assume that additionally to the hypotheses of Theorem 4.11, the manifold of parameters Z is one-dimensional. Then it is possible to compute explicitly the Novikov-Shubin numbers of the extended cohomology localized at different points of the divisor. We will see that under these assumptions the capacity of the extended cohomology is always an integer – this should be compared with examples 4.8 and 4.9, which show that the capacity may be not integral (but rational!) in the real analytic situation.

In fact, I believe that *in the real analytic situation of Theorem 4.11 the capacity of the extended cohomology of complex (4-15) is always rational*. Compare with conjecture 7.1 in [LL].

In the case of one-dimensional parameter the capacity of the extended L^2 cohomology can be expressed using the notions of *germ-complex* and *germ-cohomology*, which were introduced in [Fa2] and [FL]. These notions are defined in a more general situation when we have a family of elliptic complexes (or a finite dimensional family, like (4-14)) depending on a parameter $t \in (t_0 - \epsilon, t_0 + \epsilon)$ in a real analytic fashion. The germ complex is constructed similarly to complex (4-15), but instead of L^2 sections, one considers *the germs at t_0 of real analytic sections*. More precisely, denote by \mathcal{OE}^i the set of germs of all real analytic sections of (4-14), defined in a neighbourhood of $t = t_0$. The differential $\partial : \mathcal{E}^i \rightarrow \mathcal{E}^{i+1}$ naturally defined the map on the sections $\partial : \mathcal{OE}^i \rightarrow \mathcal{OE}^{i+1}$. This produces a complex (\mathcal{OE}, ∂) (called the germ-complex) of free modules over the ring \mathcal{O} of germs at $t = t_0$ of real analytic complex valued functions of the parameter t . We refer the reader to [Fa2, FL] for more information and applications.

The cohomology of the germ complex $H^i(\mathcal{OE}, \partial)$ is a finitely generated module over \mathcal{O} . Since \mathcal{O} is a discrete valuation ring, the module $H^i(\mathcal{OE}, \partial)$ has well defined \mathcal{O} -torsion submodule $\tau^i \subset H^i(\mathcal{OE}, \partial)$; the whole module $H^i(\mathcal{OE}, \partial)$ is in fact a direct sum of a free \mathcal{O} -module (of rank equal to the generic Betti number β^i) and τ^i . The torsion submodule is finite dimensional as a vector space over \mathbb{C} ; it has a canonical nilpotent endomorphism $\hat{t} : \tau^i \rightarrow \tau^i$ - multiplication by $t - t_0 \in \mathcal{O}$. The minimal number n such that \hat{t}^n annihilates τ^i (i.e. $\hat{t}^n \cdot \tau^i = 0$) will be called *the height of τ^i* .

Now we may formulate an addition to Theorem 4.11 in the case of one-dimensional parameter space.

4.13. Theorem. *If manifold Z in Theorem 4.11 is one-dimensional, then for any i the divisor of the extended cohomology $\mathcal{H}^i(L^2(\mathcal{E}), \partial^*)$ of complex (4-15) consists of finitely many isolated points. If t_0 denotes one of the points of the divisor, then the capacity of the extended cohomology $\mathcal{H}^i(L^2(\mathcal{E}), \partial^*)$ with respect to a measure of the form $\nu = \chi_{[t_0 - \epsilon', t_0 + \epsilon']} (t) \times \mu$ (where χ denotes the characteristic function of the indicated interval and μ is the Lebesgue measure) for small enough ϵ' equals to the height of the torsion part of the germ cohomology*

$$c_\nu(\mathcal{H}^i(L^2(\mathcal{E}), \partial^*)) = \text{height of } \tau^i.$$

In particular, this capacity is integral.

Proof. First of all, applying the recipe described in section 3.14, we realize that to compute the spectral density function of $\mathcal{H}^i(L^2(\mathcal{E}), \partial^*)$ we have to consider the self-adjoint bundle map $\partial^* \partial : \mathcal{E}^{i-1} \rightarrow \mathcal{E}^{i-1}$. Using the Rellich - Kato theorem (cf. [K], chapter 7), we may find *real analytic functions*

$$\lambda_1, \dots, \lambda_m : U \rightarrow \mathbb{R},$$

where m is the rank of the bundle \mathcal{E}^{i-1} and U is a neighbourhood of t_0 , such that for any $t \in U$ the numbers $\lambda_1(t), \dots, \lambda_m(t)$ are all the eigenvalues of $\partial^* \partial$. There can be several functions $\lambda_i(t)$, which are identically zero. We will assume that the $\lambda_i(t)$'s are numerated such that $\lambda_i(t) \equiv 0$ for $1 \leq i < m'$ and for $m' \leq i \leq m$ the functions $\lambda_i(t)$ are not identically zero.

Now, similarly to (4-4), the spectral density function $F_\nu(\lambda)$ of $\mathcal{H}^i(L^2(\mathcal{E}), \partial^*)$ is given by

$$F_\nu(\lambda) = \sum_{i=m'}^m \nu\{t \in U; \lambda_i(t) \leq \lambda^2\}.$$

Since $\lambda_i(t)$ is real analytic and non-negative around t_0 , we may write

$$\lambda_i(t) = (t - t_0)^{2k_i} \gamma_i(t), \quad i = m', \dots, m,$$

where k_i is a non-negative integer and $\gamma_i(t_0) \neq 0$. Note, that the point t_0 belongs to the divisor if and only if at least one of these numbers k_i is non-zero (by Theorem 4.11). Thus, we obtain that the points of the divisor are isolated and also, since

$$\nu\{t \in U; \lambda_i(t) \leq \lambda^2\} \sim \lambda^{k_i^{-1}}, \quad \text{for } k_i \neq 0,$$

the Novikov - Shubin number of $\mathcal{H}^i(L^2(\mathcal{E}), \partial^*)$ is $\min\{k_i^{-1}\}$. Therefore, the capacity $\mathbf{c}_\nu(\mathcal{H}^i(L^2(\mathcal{E}), \partial^*))$ is $\max\{k_i\}$.

The Rellich - Kato theorem [K], chapter 7, gives us also an orthonormal system of eigensections $s_1(t), \dots, s_m(t)$ of \mathcal{E}^{i-1} for $t \in U$ such that $\partial^* \partial s_i(t) = \lambda_i(t) s_i(t)$ and such that $s_i(t)$ are real analytic in t . For $m' \leq i \leq m$ the sections $\lambda_i^{-1/2}(t) \cdot \partial s_i(t)$ of \mathcal{E}^i are then analytic and form an orthonormal basis in the space of analytic sections of \mathcal{E}^i which belong to the image of ∂ after multiplication by a power of $t - t_0$ (compare with the definition of the germ-complex in [Fa2]). Thus, it follows that the height of torsion submodule τ^i of the cohomology of the germ-complex also equals $\max\{k_i\}$. \square

§5. Abelian extensions of categories of Hilbertian representations

In this section we mention briefly the similar construction of building the abelian extension of an additive category in the case when instead of Hilbert representations we start with a category of *Hilbertian representations*. This situation appears to be more natural from the point of view of many topological applications. For example, Hilbertian representations are important if one studies the L^2 torsion, combinatorial and analytic (as in [CFM]). Another class of examples (which seem to be important for constructing a De Rham version of the extended L^2 -cohomology, cf. §7) provide Sobolev spaces of sections of vector bundles with Hilbert fiber over a compact manifold. We will apply the results of this section in §7.

5.1. Hilbertian spaces. Recall that a *Hilbertian space* (cf. [P]) is a topological vector space \mathcal{H} which is isomorphic to a Hilbert space in the category of topological vector spaces. In other words, there exists a scalar product on \mathcal{H} such that \mathcal{H} with this scalar product is a Hilbert space with the originally given topology. Such scalar products are called *admissible*. Given one admissible scalar product $\langle \cdot, \cdot \rangle$ on \mathcal{H} , any other admissible scalar product is given by

$$\langle x, y \rangle_1 = \langle Ax, y \rangle, \quad x, y \in \mathcal{H}, \quad (5-1)$$

where $A : \mathcal{H} \rightarrow \mathcal{H}$ is an invertible positive operator $A^* = A$, $A > 0$.

Hilbertian spaces naturally appear as Sobolev spaces of vector bundles, cf. [P].

5.2. Hilbertian representations. Let \mathcal{A} be an algebra with an involution, as in 1.1. A *Hilbertian representation of \mathcal{A}* is a Hilbertian topological vector space \mathcal{H} supplied with a left action of \mathcal{A} by continuous linear maps $\mathcal{A} \rightarrow \mathcal{L}(\mathcal{H}, \mathcal{H})$ such that there exists an admissible scalar product on \mathcal{H} with respect to which the given action is a Hilbert representation, cf. 1.1. We will say that this scalar product is *admissible with respect to the given Hilbertian representation*. Again, given one such admissible scalar product, all other admissible scalar products can be constructed as in the previous paragraph, where A is any invertible positive operator *commuting with the action of \mathcal{A}* .

A morphism $f : \mathcal{H}_1 \rightarrow \mathcal{H}_2$ between two Hilbertian representations is defined in the same way as in 1.1, i.e. as a bounded linear map commuting with the action of the algebra \mathcal{A} .

5.3. Hilbertian categories. As in 1.1, we may consider a subcategory $\mathcal{C}_{\mathcal{A}}$ of the category of Hilbertian representations of a given algebra with involution \mathcal{A} .

We will say that $\mathcal{C}_{\mathcal{A}}$ is a *Hilbertian category* if any object \mathcal{H} of $\mathcal{C}_{\mathcal{A}}$ has a $\mathcal{C}_{\mathcal{A}}$ -admissible scalar product (cf. below) and $\mathcal{C}_{\mathcal{A}}$ satisfies condition (i) of section 1.2 and also condition (ii) of section 1.2 in the following edition:

- (ii*) *For any $\mathcal{H} \in \text{ob}(\mathcal{C}_{\mathcal{A}})$ the dual representation \mathcal{H}^* also is an object of $\mathcal{C}_{\mathcal{A}}$ and for any morphism $\phi : \mathcal{H}_1 \rightarrow \mathcal{H}_2$ of $\mathcal{C}_{\mathcal{A}}$ the adjoint operator $\phi^* : \mathcal{H}_2^* \rightarrow \mathcal{H}_1^*$ is also a morphism of $\mathcal{C}_{\mathcal{A}}$.*

Note, that the dual representation \mathcal{H}^* is defined as the space of all bounded anti-linear functionals $\phi : \mathcal{H} \rightarrow \mathbb{C}$, $\phi(\lambda x) = \bar{\lambda}\phi(x)$ for $x \in \mathcal{H}$ and $\lambda \in \mathbb{C}$. The \mathcal{A} -module structure on \mathcal{H}^* is given by $(a\phi)(x) = \phi(a^*x)$ for $a \in \mathcal{A}$ and $x \in \mathcal{H}$.

If $\mathcal{C}_{\mathcal{A}}$ is a category of Hilbertian representations and $\mathcal{H} \in \text{ob}(\mathcal{C}_{\mathcal{A}})$, then we can specify naturally a class of admissible scalar products on \mathcal{H} by declaring an admissible scalar product $\langle \cdot, \cdot \rangle$ on \mathcal{H} to be $\mathcal{C}_{\mathcal{A}}$ -admissible if the isomorphism $\mathcal{H} \rightarrow \mathcal{H}^*$ determined by $\langle \cdot, \cdot \rangle$ belongs to $\mathcal{C}_{\mathcal{A}}$.

If $\langle \cdot, \cdot \rangle$ and $\langle \cdot, \cdot \rangle_1$ are two $\mathcal{C}_{\mathcal{A}}$ -admissible scalar products on \mathcal{H} then the corresponding operator A (which appears in (5-1)) and its inverse A^{-1} are morphisms of $\mathcal{C}_{\mathcal{A}}$.

5.4. Abelian extension of a Hilbertian category. Repeating construction described in section 1.3, for any Hilbertian category $\mathcal{C}_{\mathcal{A}}$ one obtains an extended abelian category $\mathcal{E}(\mathcal{C}_{\mathcal{A}})$ containing $\mathcal{C}_{\mathcal{A}}$ as a full subcategory. We will call $\mathcal{E}(\mathcal{C}_{\mathcal{A}})$ the *abelian extension of $\mathcal{C}_{\mathcal{A}}$* . Any object of the extended category $\mathcal{E}(\mathcal{C}_{\mathcal{A}})$ is represented as a morphism $(\alpha : A' \rightarrow A)$ in $\mathcal{C}_{\mathcal{A}}$. As in section 1.8, there is a full embedding $\mathcal{C}_{\mathcal{A}} \rightarrow \mathcal{E}(\mathcal{C}_{\mathcal{A}})$;

similarly to Proposition 1.9 we have that any object of the original category $\mathcal{C}_{\mathcal{A}}$ is projective in $\mathcal{E}(\mathcal{C}_{\mathcal{A}})$ and conversely, any projective object of $\mathcal{E}(\mathcal{C}_{\mathcal{A}})$ is isomorphic to an object of $\mathcal{C}_{\mathcal{A}}$ inside $\mathcal{E}(\mathcal{C}_{\mathcal{A}})$.

It is easy to check that all the statements and the arguments of subsections 1.1 - 1.11 equally apply to the Hilbertian situation as well.

5.5. Hilbertian von Neumann categories. A Hilbertian category $\mathcal{C}_{\mathcal{A}}$ will be called *von Neumann category* if it satisfies condition (v) of section 2.1.

An object \mathcal{H} of a Hilbertian von Neumann category will be called *finite* if the only closed $\mathcal{C}_{\mathcal{A}}$ -submodule $\mathcal{H}_1 \subset \mathcal{H}$ which is isomorphic to \mathcal{H} in $\mathcal{C}_{\mathcal{A}}$ is $\mathcal{H}_1 = \mathcal{H}$.

A Hilbertian von Neumann category will be called *finite* iff all its objects are finite.

Examples of Hilbertian von Neumann categories can be obtained from examples considered in §2 by forgetting the scalar product.

5.6. Torsion objects of the extended category. Suppose that $\mathcal{C}_{\mathcal{A}}$ is a finite Hilbertian von Neumann category. Then similarly to §3, we may define the notion of a torsion object. Denoting by $\mathcal{T}(\mathcal{C}_{\mathcal{A}})$ the full subcategory of the extended category $\mathcal{E}(\mathcal{C}_{\mathcal{A}})$ generated by torsion objects, we will have (similarly to Corollary 3.3) that the torsion subcategory $\mathcal{T}(\mathcal{C}_{\mathcal{A}})$ is *an abelian subcategory* of the extended category.

5.7. The main distinction between this Hilbertian version and the situation of §2 is that in the Hilbertian case for any $\mathcal{H} \in \text{ob}(\mathcal{C}_{\mathcal{A}})$ we do not have a fixed $*$ -operator on $\text{Hom}_{\mathcal{C}_{\mathcal{A}}}(\mathcal{H}, \mathcal{H})$, and so it is not a von Neumann algebra. Instead, we have many $*$ -operators (each corresponding to a choice of an admissible scalar product on \mathcal{H}), and considered with any of these involutions the algebra $\text{Hom}_{\mathcal{C}_{\mathcal{A}}}(\mathcal{H}, \mathcal{H})$ becomes a von Neumann algebra. We observe that the notion of positivity of an element of $\text{Hom}_{\mathcal{C}_{\mathcal{A}}}(\mathcal{H}, \mathcal{H})$ also depends on the choice of an admissible scalar product on \mathcal{H} .

5.8. Traces. A *trace on a Hilbertian von Neumann category* $\mathcal{C}_{\mathcal{A}}$ is a function tr which assigns to each object $\mathcal{H} \in \text{ob}(\mathcal{C}_{\mathcal{A}})$ a linear functional

$$\text{tr}_{\mathcal{H}} : \text{Hom}_{\mathcal{C}_{\mathcal{A}}}(\mathcal{H}, \mathcal{H}) \rightarrow \mathbb{C} \quad (5-2)$$

such that for any pair of representations $\mathcal{H}_1, \mathcal{H}_2 \in \text{ob}(\mathcal{C}_{\mathcal{A}})$ the corresponding traces $\text{tr}_{\mathcal{H}_1}$ and $\text{tr}_{\mathcal{H}_2}$ are compatible in the following sense: if $f \in \text{Hom}_{\mathcal{C}_{\mathcal{A}}}(\mathcal{H}_1, \mathcal{H}_2)$ and $g \in \text{Hom}_{\mathcal{C}_{\mathcal{A}}}(\mathcal{H}_2, \mathcal{H}_1)$ then

$$\text{tr}_{\mathcal{H}_1}(gf) = \text{tr}_{\mathcal{H}_2}(fg). \quad (5-3)$$

We may call trace tr *non-negative* if $\text{tr}_{\mathcal{H}}(e)$ is real and non-negative for any idempotent $e \in \text{Hom}_{\mathcal{C}_{\mathcal{A}}}(\mathcal{H}, \mathcal{H})$, $e^2 = e$.

5.9. von Neumann dimension and the Novikov-Shubin invariants. Any trace on a von Neumann category defines a *von Neumann dimension function* (as in 2.8); the dimension is always non-negative and real valued iff the trace tr is non-negative (in the above sense).

Using a non-negative trace on $\mathcal{C}_{\mathcal{A}}$ one defines for any torsion object \mathcal{X} of the extended category $\mathcal{E}(\mathcal{C}_{\mathcal{A}})$ the *spectral density function* (as in 3.7) and then the *Novikov-Shubin number* $\text{ns}(\mathcal{X})$ or the *capacity* $\text{c}(\mathcal{X})$ (as in 3.9).

Hilbertian versions of Proposition 3.10 and Corollary 3.11 hold.

The following Lemma justifies our definition of the trace on Hilbertian category.

5.10. Lemma. *Suppose that \mathcal{C}_A is a Hilbertian category and $\mathcal{H} \in \text{ob}(\mathcal{C}_A)$. Let $\text{tr}_{\mathcal{H}} : \text{Hom}_{\mathcal{C}_A}(\mathcal{H}, \mathcal{H}) \rightarrow \mathbb{C}$ be a linear functional which is traceful, i.e. $\text{tr}_{\mathcal{H}}(fg) = \text{tr}_{\mathcal{H}}(gf)$ for any $f, g \in \text{Hom}_{\mathcal{C}_A}(\mathcal{H}, \mathcal{H})$. Suppose that $\text{tr}_{\mathcal{H}}$ is non-negative and normal with respect to one choice of a \mathcal{C}_A -admissible scalar product on \mathcal{H} (i.e. it assumes real non-negative values on the positive cone $\text{Hom}_{\mathcal{C}_A}(\mathcal{H}, \mathcal{H})^+$ taken with respect to the given $*$ -operator). Then it is non-negative and normal with respect to any other choice of a \mathcal{C}_A -admissible scalar product on \mathcal{H} .*

Proof. This follows from [Di], part I, chapter 6, Proposition 1 using the fact that any idempotent in a von Neumann algebra is conjugate to a projection. \square

§6. Extended L^2 cohomology of cell complexes

Our aim in this section is to construct homology and cohomology theories on the category of finite polyhedra with values in the extended abelian category $\mathcal{E}(\mathcal{C}_A)$. Then we compute the extended cohomology of mapping tori.

We will start with some very general remarks, which will be then used in the main construction, cf. 6.5. Our goal is to describe *the coefficient systems*.

6.1. Let \mathcal{C} be an additive category. The objects of \mathcal{C} will be denoted \mathcal{X}, \mathcal{Y} etc., and the set of morphisms between \mathcal{X} and \mathcal{Y} will be denoted by $\text{hom}_{\mathcal{C}}(\mathcal{X}, \mathcal{Y})$.

Let Λ be a ring with a unit. We will assume that Λ has IBN (invariance of the basis number) property, cf. [C]. For our applications we need only the case $\Lambda = \mathbb{C}[\pi]$, where π is a discrete group; then this property is automatically satisfied, cf. [C].

Definition. A *structure of left Λ -module* on an object $\mathcal{X} \in \text{ob}(\mathcal{C})$ consists in specifying a ring homomorphism $\rho : \Lambda \rightarrow \text{hom}_{\mathcal{C}}(\mathcal{X}, \mathcal{X})$. Similarly, a *structure of right Λ -module* on \mathcal{X} is given by a ring homomorphism $\rho : \Lambda^{op} \rightarrow \text{hom}_{\mathcal{C}}(\mathcal{X}, \mathcal{X})$, where Λ^{op} is the opposite ring of Λ .

If one of these structures is chosen, we will say that \mathcal{X} is a *left (right) Λ -module* in \mathcal{C} . We will usually omit ρ from the notation. The left (right) Λ -modules in the category of abelian groups correspond to the usual notion of left (right) Λ -module.

If \mathcal{X} and \mathcal{Y} are two left (right) Λ -modules in \mathcal{C} then a morphism $f \in \text{hom}_{\mathcal{C}}(\mathcal{X}, \mathcal{Y})$ is called a *Λ -homomorphism* if for any $a \in \Lambda$ the following diagram commutes

$$\begin{array}{ccc} \mathcal{X} & \xrightarrow{f} & \mathcal{Y} \\ \rho(a) \downarrow & & \downarrow \rho(a) \\ \mathcal{X} & \xrightarrow{f} & \mathcal{Y} \end{array} \quad (6-1)$$

Thus we have a category of left Λ -modules in a given additive category \mathcal{C} , which we will denote $\Lambda\text{-mod-}\mathcal{C}$; the similar category of right Λ -modules in \mathcal{C} will be denoted $\mathcal{C}\text{-mod-}\Lambda$.

If \mathcal{X} is a left Λ -module in \mathcal{C} then for any $\mathcal{Y} \in \text{ob}(\mathcal{C})$ the set of morphisms $\text{hom}_{\mathcal{C}}(\mathcal{Y}, \mathcal{X})$ has a natural structure of a left Λ -module, where for $a \in \Lambda$ and $f \in \text{hom}_{\mathcal{C}}(\mathcal{Y}, \mathcal{X})$ we define

$$a \cdot f = \rho(a) \circ f : \mathcal{Y} \rightarrow \mathcal{X}. \quad (6-2)$$

Thus, if \mathcal{X} is a Λ -module in \mathcal{C} , the functor $\text{hom}_{\mathcal{C}}(\cdot, \mathcal{X})$ assumes its values in the category of left Λ -modules.

Let F be a free finitely generated left Λ -module (in the usual sense) and let \mathcal{X} be a left Λ -module in an additive category \mathcal{C} . We are now going to define

$$\mathfrak{hom}_\Lambda(F, \mathcal{X}), \quad (6-3)$$

which will be an object of category \mathcal{C} . It will depend functorially on both F and \mathcal{X} , as stated in the following Proposition.

6.2. Proposition. *There exists a bifunctor*

$$\mathfrak{hom}_\Lambda(\cdot, \cdot) : \{\text{free f.g. left } \Lambda\text{-modules}\} \times (\Lambda\text{-mod-}\mathcal{C}) \rightarrow \mathcal{C}, \quad (6-4)$$

which is contravariant with respect to the first variable and covariant with respect to the second variable and such that there is a natural isomorphism

$$\text{hom}_\mathcal{C}(\mathcal{Y}, \mathfrak{hom}_\Lambda(F, \mathcal{X})) \simeq \text{Hom}_\Lambda(F, \text{hom}_\mathcal{C}(\mathcal{Y}, \mathcal{X})) \quad (6-5)$$

where $\mathcal{Y} \in \text{ob}(\mathcal{C})$, $\mathcal{X} \in \text{ob}(\Lambda\text{-mod-}\mathcal{C})$ and F is a free finitely generated left Λ -module. If the category \mathcal{C} is abelian, then (6-4) is exact as a functor of \mathcal{X} .

Proof. For each free f.g. left Λ -module F choose a base $e = (e_1, e_2, \dots, e_n)$, where $n = \text{rank } F$. Here we should assume that we are dealing with a small category of free finitely generated left Λ -modules and then such choice is possible. The functor which we will construct, considered up to isomorphism, is independent on this choice.

Define $\mathfrak{hom}_\Lambda(F, \mathcal{X})$ as $\mathcal{X} \oplus \mathcal{X} \oplus \dots \oplus \mathcal{X}$ (n times). It obviously behaves functorially with respect to \mathcal{X} and is exact if \mathcal{C} is abelian. If F' is another free f.g. left Λ -module with the base $e' = (e'_1, e'_2, \dots, e'_m)$, where $m = \text{rank } F'$, then any Λ -homomorphism $\phi : F \rightarrow F'$ is represented in the chosen bases by a matrix (a_{ij}) , where $\phi(e_i) = \sum_{j=1}^m a_{ij}e'_j$, with $a_{ij} \in \Lambda$. Then we define $\phi^* : \mathfrak{hom}_\Lambda(F', \mathcal{X}) \rightarrow \mathfrak{hom}_\Lambda(F, \mathcal{X})$ as the morphism which maps the j -th copy of \mathcal{X} to i -th copy of \mathcal{X} by $\rho(a_{ij})$. \square

6.3. Change of rings. Suppose that $\phi : \Lambda \rightarrow \Lambda'$ is a ring homomorphism. Then any left (right) Λ' -module \mathcal{X} in \mathcal{C} determines via ϕ a left (right) Λ -module in \mathcal{C} which we denote $\phi^*(\mathcal{X})$.

Also, suppose that F and F' are free finitely generated modules over Λ and Λ' correspondingly and let $f : F \rightarrow F'$ be a Λ -homomorphism (via ϕ). Then f induces canonically the morphism

$$f^* : \mathfrak{hom}_{\Lambda'}(F', \mathcal{X}) \rightarrow \mathfrak{hom}_\Lambda(F, \phi^*(\mathcal{X})). \quad (6-6)$$

It is defined by using the definition of $\mathfrak{hom}_{\Lambda'}(F', \mathcal{X})$ above: free basis of F and F' give a representation of the map f by a matrix with entries in Λ' which then may act (using the module structure) on the direct sums of copies of \mathcal{X} similarly to the arguments above.

We will mention here one situation when homomorphism (6-6) is an isomorphism; we will refer to it later. Suppose that \mathcal{X} is a left Λ -module in \mathcal{C} and $\rho : \Lambda \rightarrow \text{hom}_\mathcal{C}(\mathcal{X}, \mathcal{X})$ is the action homomorphism. Its kernel $\ker(\rho)$ is an ideal in Λ . Let Λ' be the factor-ring $\Lambda/\ker(\rho)$. \mathcal{X} is well defined as a Λ' -module. For any free finitely generated left Λ -module F consider the morphism $f : F \rightarrow F' = F/\ker(\rho)F$. Then the corresponding morphism (6-6) is an isomorphism.

6.4. Tensor products. We will consider here a similar construction in the covariant modification.

Suppose that F is a *left* free f.g. Λ -module and \mathcal{X} is a *right* Λ -module in an additive category \mathcal{C} . Similarly to Proposition 6.2, one defines the tensor product $\mathcal{X} \tilde{\otimes}_{\Lambda} F$, giving a bifunctor

$$\tilde{\otimes}_{\Lambda} : \mathcal{C}\text{-mod-}\Lambda \times \{\text{free f.g. left } \Lambda\text{-modules}\} \rightarrow \mathcal{C}, \quad (6-7)$$

which is covariant with respect to both variables. For any choice of a free basis of F , the tensor product $\mathcal{X} \tilde{\otimes}_{\Lambda} F$ can be identified with the direct sum $\mathcal{X} \oplus \cdots \oplus \mathcal{X}$ (n copies), where n is the rank of F .

If $\mathcal{Y} \in \text{ob}(\mathcal{C})$, then the set $\text{hom}_{\mathcal{C}}(\mathcal{Y}, \mathcal{X})$ has a structure of a right Λ -module (in the usual sense) and there is natural isomorphism

$$\text{hom}_{\mathcal{C}}(\mathcal{Y}, \mathcal{X} \tilde{\otimes}_{\Lambda} F) \simeq \text{hom}_{\mathcal{C}}(\mathcal{Y}, \mathcal{X}) \otimes_{\Lambda} F, \quad (6-8)$$

similar to (6-5).

6.5. The basic construction. Now we may define extended homology and cohomology of cell complexes. The construction in the form presented here, generalizes the construction of [Fa1], §6.

Let K be a connected CW complex with a base point. We will assume that K has finitely many cells in each dimension. Denote by $\pi = \pi_1(K)$ the fundamental group. Let \tilde{K} denotes the universal cover of K . We will suppose that a base point has been chosen in \tilde{K} , lying above the base point of K . Then the group of covering transformations of \tilde{K} can be identified with π . Consider the chain complex $C_*(\tilde{K})$ constructed using the cells of the universal covering \tilde{K} , which are the lifts of the cells of K . The fundamental group π acts on $C_*(\tilde{K})$ from the left and in each dimension i the chain module $C_i(\tilde{K})$ is a free $\mathbb{C}[\pi]$ -module with the number of generators equal to the number of i -dimensional cells in K .

Let \mathcal{A} be a $*$ -algebra and let $\mathcal{C}_{\mathcal{A}}$ be a Hilbert category of $*$ -representations of \mathcal{A} , cf. 1.2. (Later we will impose an additional assumption that $\mathcal{C}_{\mathcal{A}}$ is a finite von Neumann category, but most of the constructions hold in this more general context.) Let $\mathcal{E}(\mathcal{C}_{\mathcal{A}})$ denote the extended abelian category of $\mathcal{C}_{\mathcal{A}}$, cf. §1.

Assume that we are given a left $\mathbb{C}[\pi]$ -module \mathcal{M} in $\mathcal{E}(\mathcal{C}_{\mathcal{A}})$, as defined in section 6.1. We will consider some examples later in this section. The module \mathcal{M} will play role of *a coefficient system*. Let us emphasize that in our present approach \mathcal{M} may equally be projective, torsion, or a combination of a torsion module and a projective. This generality may look superfluous, but (as we are going to show in another place) only having the opportunity to use torsion Hilbert spaces as coefficient systems one may generalize naturally and fully many classical techniques, for example, the spectral sequences.

Applying the functor $\mathfrak{hom}_{\Lambda}(\cdot, \mathcal{M})$, where $\Lambda = \mathbb{C}[\pi]$, to the chain complex $C_i(\tilde{K})$, we obtain the following cochain complex $\mathfrak{hom}_{\Lambda}(C_*(\tilde{K}), \mathcal{M})$ in $\mathcal{E}(\mathcal{C}_{\mathcal{A}})$:

$$\cdots \leftarrow \mathfrak{hom}_{\Lambda}(C_{i+1}(\tilde{K}), \mathcal{M}) \leftarrow \mathfrak{hom}_{\Lambda}(C_i(\tilde{K}), \mathcal{M}) \leftarrow \mathfrak{hom}_{\Lambda}(C_{i-1}(\tilde{K}), \mathcal{M}) \leftarrow \cdots \quad (6-9)$$

The cohomology of this complex, understood as an object of the extended category $\mathcal{E}(\mathcal{C}_A)$, will be denoted

$$\mathcal{H}^i(K, \mathcal{M}) = H^i(\mathfrak{hom}_\Lambda(C_*(\tilde{K}), \mathcal{M})); \quad (6-10)$$

it will be called *extended cohomology of K with coefficients in \mathcal{M}* .

Intuitively, we may view the cochains $c \in \mathfrak{hom}_\Lambda(C_*(\tilde{K}), \mathcal{M})$ as functions which assign to cells of \tilde{K} "elements of \mathcal{M} " such that

$$c(ge) = \rho(g)(c(e)) \quad (6-11)$$

for all cells e of \tilde{K} and all $g \in \pi$. Here $\rho : \Lambda \rightarrow \text{End}_{\mathcal{E}(\mathcal{C}_A)}(\mathcal{M})$ is the module action.

Now we will define the similar notion of extended homology. Here we will assume that we are given a *right* $\mathbb{C}[\pi]$ -module \mathcal{M} in the extended category $\mathcal{E}(\mathcal{C}_A)$. We define

$$\mathcal{H}_i(K, \mathcal{M}) = H_i(\mathcal{M} \tilde{\otimes}_\Lambda C_*(\tilde{K})), \quad (6-12)$$

the *extended homology of K with coefficients in \mathcal{M}* . Here $\tilde{\otimes}_\Lambda$ denotes the tensor product introduced in section 6.4.

Note that the extended homology and cohomology are objects of the extended category and may have non-trivial torsion parts even if the module \mathcal{M} was projective.

6.6. We will consider here some simple properties of the extended cohomology considered as a functor of K . Similar properties are valid for the extended homology.

Let $f : K \rightarrow K'$ be a continuous map preserving the base points between CW complexes having finitely many cells in each dimension. It induces a homomorphism $f_* : \pi \rightarrow \pi'$, where $\pi = \pi(K)$ and $\pi' = \pi(K')$ are the fundamental groups. If \mathcal{M} is a $\mathbb{C}[\pi']$ -module in $\mathcal{E}(\mathcal{C}_A)$, then the homomorphism f_* allows to view \mathcal{M} as a $\mathbb{C}[\pi]$ -module in category $\mathcal{E}(\mathcal{C}_A)$, which we will denote $f^*\mathcal{M}$. Now, the map f lifts uniquely to a π -equivariant map $\tilde{f} : \tilde{K} \rightarrow \tilde{K}'$ preserving the base points. \tilde{f} induces a chain map $C_*(\tilde{f}) : C_*(\tilde{K}) \rightarrow C_*(\tilde{K}')$. Applying to the last map the change of rings morphism (6-6), we obtain a chain map

$$\mathfrak{hom}_{\Lambda'}(C_*(\tilde{K}'), \mathcal{M}) \rightarrow \mathfrak{hom}_\Lambda(C_*(\tilde{K}), f^*\mathcal{M}); \quad (6-13)$$

it induces the morphism of the extended cohomology

$$f^* : \mathcal{H}^*(K', \mathcal{M}) \rightarrow \mathcal{H}^*(K, f^*\mathcal{M}). \quad (6-14)$$

The induced morphism (6-14) depends only on the homotopy class of f , since the chain homotopy class of the map $C_*(\tilde{f})$ (as it is well known) depends only on the homotopy class of f .

6.7. Corollary. *The extended cohomology $\mathcal{H}^*(K, \mathcal{M})$ is a homotopy invariant of the CW complex K .*

If the category \mathcal{C}_A is a finite von Neumann category, then to any trace on \mathcal{C}_A we associate the Novikov - Shubin numbers (cf. 3.9) in order to measure the size of the torsion part of the extended homology and cohomology. This method gives *homotopy invariants* of K by Corollary 6.7 above - this fact was proven first by M.Gromov and M.Shubin [GS].

6.8. Mayer - Vietoris sequence. *Suppose that K is a connected CW complex with base point having finitely many cells in each dimension and \mathcal{M} is a $\mathbb{C}[\pi]$ -module in $\mathcal{E}(\mathcal{C}_{\mathcal{A}})$, where $\pi = \pi(K)$. Let K_1 and K_2 be two connected subcomplexes containing the base point such that $K = K_1 \cup K_2$ and the CW complex $K_0 = K_1 \cap K_2$ is connected. Denote $\mathcal{M}_i = \mathcal{M}|_{K_i}$, where $i = 0, 1, 2$. Then the following sequence in category $\mathcal{E}(\mathcal{C}_{\mathcal{A}})$ is exact*

$$\cdots \rightarrow \mathcal{H}^i(K, \mathcal{M}) \rightarrow \mathcal{H}^i(K_1, \mathcal{M}_1) \oplus \mathcal{H}^i(K_2, \mathcal{M}_2) \rightarrow \mathcal{H}^i(K_0, \mathcal{M}_0) \xrightarrow{\delta} \mathcal{H}^{i+1}(K, \mathcal{M}) \cdots \quad (6-15)$$

The morphisms of this exact sequence (except δ) are induced by the inclusions as usual.

The proof repeats the standard arguments and will be skipped.

Now we will summarize some properties of the extended cohomology as function of \mathcal{M} .

6.9. Proposition. *The extended cohomology $\mathcal{H}^i(K, \mathcal{M})$ is a covariant functor of \mathcal{M} . For any exact sequence of $\mathbb{C}[\pi]$ -modules in category $\mathcal{E}(\mathcal{C}_{\mathcal{A}})$*

$$0 \rightarrow \mathcal{M}' \rightarrow \mathcal{M} \rightarrow \mathcal{M}'' \rightarrow 0 \quad (6-16)$$

there is a natural long exact sequence

$$\cdots \mathcal{H}^i(K, \mathcal{M}') \rightarrow \mathcal{H}^i(K, \mathcal{M}) \rightarrow \mathcal{H}^i(K, \mathcal{M}'') \rightarrow \mathcal{H}^{i+1}(K, \mathcal{M}') \rightarrow \cdots \quad (6-17)$$

in $\mathcal{E}(\mathcal{C}_{\mathcal{A}})$. If $\mathcal{C}_{\mathcal{A}}$ is a finite von Neumann category (cf. §2) and \mathcal{M} is torsion (as an object of $\mathcal{E}(\mathcal{C}_{\mathcal{A}})$) then all extended cohomology $\mathcal{H}^(K, \mathcal{M})$ is torsion.*

Proof. Since the functor $\mathfrak{hom}_{\Lambda}(F, \mathcal{M})$ is exact with respect to \mathcal{M} (cf. Proposition 6.2), we obtain the following exact sequence of cochain complexes

$$0 \rightarrow \mathfrak{hom}_{\Lambda}(C_*(\tilde{K}), \mathcal{M}') \rightarrow \mathfrak{hom}_{\Lambda}(C_*(\tilde{K}), \mathcal{M}) \rightarrow \mathfrak{hom}_{\Lambda}(C_*(\tilde{K}), \mathcal{M}'') \rightarrow 0, \quad (6-18)$$

which produces by the general laws of homological algebra the exact sequence (6-17). If \mathcal{M} is a torsion object then all objects $\mathfrak{hom}_{\Lambda}(C_i(\tilde{K}), \mathcal{M})$ are torsion and hence the cohomology is torsion (by corollary 3.3). \square

Now we are going to discuss some examples of $\mathbb{C}[\pi]$ -modules in the abelian extensions of Hilbert and von Neumann categories, and the corresponding extended cohomology of polyhedra.

6.10. Example: The regular representation. The simplest example of a $\mathbb{C}[\pi]$ -module in $\mathcal{E}(\mathcal{C}_{\mathcal{A}})$ can be obtained if we are given a representation of π acting on a Hilbert space \mathcal{H} , such that for any $g \in \pi$ the morphism $\mathcal{H} \rightarrow \mathcal{H}$ of multiplication by g , belongs to a Hilbert category $\mathcal{C}_{\mathcal{A}}$. Then \mathcal{H} is a $\mathbb{C}[\pi]$ -module in $\mathcal{E}(\mathcal{C}_{\mathcal{A}})$.

The most familiar example of this sort is given by the regular representation $\ell^2(\pi)$ with π acting from the left by translations. The corresponding category $\mathcal{C}_{\mathcal{A}}$ in this case is the finite von Neumann category described in example 5 of section 2.6. The algebra \mathcal{A} is $\mathcal{N}(\pi)^{\bullet}$, i.e. the algebra opposite to $\mathcal{N}(\pi)$, the von Neumann algebra of π , cf. 2.6, example 5.

The cochains in this situation have very transparent geometric meaning. Namely, any cochain $c \in \mathfrak{hom}_\Lambda(C_*(\tilde{K}), \ell^2(\pi))$ is a function which assigns an element $c(e) \in \ell^2(\pi)$ to any cell e of the universal covering \tilde{K} such that $c(ge) = gc(e)$ holds (cf. (6-11)) for any $g \in \pi$. Writing $c(e) = \sum_{g \in \pi} \alpha_g \cdot g$, where $\alpha_g \in \mathbb{C}$ satisfy $\sum |\alpha_g|^2 < \infty$, we may consider a cochain on \tilde{K} with values in \mathbb{C} , where $h(e) = \alpha_1$ (here 1 denotes the unit element of π). Then $h(g^{-1}e) = \alpha_g$ and we obtain that h is a \mathbb{C} -valued cochain satisfying the L^2 -condition:

$$\sum_e |h(e)|^2 < \infty. \quad (6-19)$$

In the last formula e runs over all cells of \tilde{K} . Using the same arguments in the opposite way, we see that any cochain on \tilde{K} with values in \mathbb{C} which satisfies L^2 -condition comes from a cochain with values in $\ell^2(\pi)$ considered as a $\mathbb{C}[\pi]$ -module. Comparing also the boundary maps shows that in this case the complex $\mathfrak{hom}_\Lambda(C_*(\tilde{K}), \ell^2(\pi))$ can be identified with the complex built on the \mathbb{C} -valued cochain with L^2 condition.

A slightly more general example is given by the module $V \otimes \ell^2(\pi)$, where V is a finite dimensional Hilbert representation of π . The group π acts diagonally on the tensor product above (which is understood over \mathbb{C}); the corresponding von Neumann category is described in example 6 of section 2.6. This example may also be described geometrically similar to the previous discussion as the cochain complex formed by chains h on \tilde{K} with values in V , which satisfy the following L^2 -condition $\sum_{e \in \tilde{K}} |h(e)|^2 < \infty$. The last sum is taken over all cells of \tilde{K} .

6.11. Here is a remark concerning building of $\mathbb{C}[\pi]$ modules in the extended category.

Generally, if we have two $\mathbb{C}[\pi]$ -modules \mathcal{H}_0 and \mathcal{H}_1 in $\mathcal{C}_\mathcal{A}$ and a $\mathbb{C}[\pi]$ -morphism $\alpha : \mathcal{H}_0 \rightarrow \mathcal{H}_1$ then $\mathcal{M} = (\alpha : \mathcal{H}_0 \rightarrow \mathcal{H}_1)$ represents a $\mathbb{C}[\pi]$ -module in $\mathcal{E}(\mathcal{C}_\mathcal{A})$. Not all examples are of this form.

For instance, consider the torsion object $\mathcal{X}_{\theta, \nu} = (\alpha : L^2(S^1) \rightarrow L^2(S^1))$ as in example 3.12. Let the function $f \in L^\infty(S^1)$ be given by $f(z) = \sqrt{1 + \alpha(z)}$. Then the morphism $m_f : \mathcal{X}_{\theta, \nu} \rightarrow \mathcal{X}_{\theta, \nu}$ (here m_f denotes the operator of multiplication by f) represented by the diagram

$$\begin{array}{ccc} (\alpha : \mathcal{X}_{\theta, \nu} \rightarrow \mathcal{X}_{\theta, \nu}) & & \\ m_f \downarrow & & \downarrow m_f \\ (\alpha : \mathcal{X}_{\theta, \nu} \rightarrow \mathcal{X}_{\theta, \nu}) & & \end{array}$$

defines an action of the group \mathbb{Z}_2 on $\mathcal{X}_{\theta, \nu}$. Although it is only a Z action on $L^2(S^1)$.

6.12. Example: Extended cohomology of a mapping torus. For illustrative purposes and because of many applications, we will compute here the extended cohomology of a mapping torus.

Suppose that Y is a finite polyhedron with a base point y_0 and let $\phi : Y \rightarrow Y$ be a homeomorphism with $\phi(y_0) = y_0$. Consider the mapping torus $K = K_\phi = Y \times [0, 1] / \sim$, where $(y, 1) \sim (\phi(y), 0)$ for all $y \in Y$. We will consider $(y_0, 0)$ as the base point of K .

We have the following exact sequence of the fundamental groups

$$0 \rightarrow \pi(Y) \rightarrow \pi(K) \rightarrow \mathbb{Z} \rightarrow 0. \quad (6-20)$$

This sequence clearly splits. The natural splitting is given by sending the generator of \mathbb{Z} to the class of the loop (y_0, t) , where $t \in [0, 1]$; the class of this loop in $\pi(K)$ will be denoted τ . We see that $\pi(K)$ is a semi-direct product; the following relation holds in $\pi(K)$:

$$\tau g \tau^{-1} = \phi_*(g) \quad (6-21)$$

for all $g \in \pi(Y)$, where $\phi_* : \pi(Y) \rightarrow \pi(Y)$ is the automorphism induced by ϕ .

The universal covering \tilde{K} of K can be identified with the product $\tilde{Y} \times \mathbb{R}$, where \tilde{Y} is the universal covering of Y . To describe the action of the fundamental group $\pi(K)$ on $\tilde{Y} \times \mathbb{R}$ we have to fix a base point in \tilde{Y} (above y_0) and then there is a unique lift $\tilde{\phi} : \tilde{Y} \rightarrow \tilde{Y}$ of the homeomorphism ϕ , preserving the base points. Such lift $\tilde{\phi}$ is *equivariant* in the following sense:

$$\tilde{\phi}(g \cdot \tilde{y}) = \phi_*(g) \cdot \tilde{\phi}(\tilde{y}). \quad (6-22)$$

Any element $g \in \pi(Y)$ acts on $\tilde{K} = \tilde{Y} \times \mathbb{R}$ by $g \cdot (\tilde{y}, t) = (g \cdot \tilde{y}, t)$ for all $\tilde{y} \in \tilde{Y}$, $t \in \mathbb{R}$, and the action of the element τ is given by $\tau \cdot (\tilde{y}, t) = (\tilde{\phi}(\tilde{y}), t - 1)$.

The cell structure of \tilde{K} is as follows. For any n -dimensional cell $e \subset Y$, the universal covering \tilde{K} has n -dimensional cells of the form $\tau^i(ge)$, where $i \in \mathbb{Z}$, $g \in \pi(Y)$, and $(n + 1)$ -dimensional cells of the form $\tau^i(ge \times (0, 1))$, where $i \in \mathbb{Z}$, and $g \in \pi(Y)$.

This allows to describe completely the chain complex $C_*(\tilde{K})$ as a complex of left Λ -modules, where $\Lambda = \mathbb{C}[\pi(K)]$. Denote $\Lambda_Y = \mathbb{C}[\pi(Y)]$, which will be understood as a subring of Λ via (6-20). Then the morphism of complexes of left Λ -modules

$$\tau^{-1} \otimes C_*(\tilde{\phi}) : \Lambda \otimes_{\Lambda_Y} C_*(\tilde{Y}) \rightarrow \Lambda \otimes_{\Lambda_Y} C_*(\tilde{Y}), \quad (6-23)$$

which is given by the formula

$$\lambda \otimes c \mapsto \lambda \tau^{-1} \otimes C_*(\tilde{\phi})(c), \quad \lambda \in \Lambda, \quad c \in C_*(\tilde{Y}), \quad (6-24)$$

is well defined, and is a chain morphism of complexes over Λ . Here $C_*(\phi) : C_*(\tilde{Y}) \rightarrow C_*(\tilde{Y})$ denotes the chain map induced by $\tilde{\phi}$. Comparing with the description of the cell structure of \tilde{K} in the previous paragraph, one obtains that *the chain complex $C_*(\tilde{K})$ coincides with the mapping cone of the following chain map*

$$\text{id} - \tau^{-1} \otimes C_*(\tilde{\phi}) : \Lambda \otimes_{\Lambda_Y} C_*(\tilde{Y}) \rightarrow \Lambda \otimes_{\Lambda_Y} C_*(\tilde{Y}). \quad (6-25)$$

Now, to compute the extended cohomology of K (according to the basic construction of section 6.5) we have to apply the functor $\mathfrak{hom}_\Lambda(\cdot, \mathcal{M})$ to the mapping cone of (6-25). Here \mathcal{M} denotes a Λ -module in the extended category $\mathcal{E}(\mathcal{C}_A)$, which has to be chosen as the coefficient system. We obtain a representation of the complex $\mathfrak{hom}_\Lambda(C_*(\tilde{K}), \mathcal{M})$ also as a mapping cone (with a slight shift of the dimensions) of a morphism of cochain complexes

$$\mathfrak{hom}_{\Lambda_Y}(C_*(\tilde{Y}), \mathcal{M}|_Y) \rightarrow \mathfrak{hom}_{\Lambda_Y}(C_*(\tilde{Y}), \mathcal{M}|_Y) \quad (6-26)$$

in $\mathcal{E}(\mathcal{C}_A)$. Here $\mathcal{M}|_Y$ denotes \mathcal{M} but considered only as a left Λ_Y -module.

Observe that defining \mathcal{M} as a module over Λ is equivalent to defining it as a module over Λ_Y and specifying an isomorphism of left Λ_Y -modules

$$\tau : \mathcal{M} \rightarrow \phi^* \mathcal{M}, \quad (6-27)$$

which is given by multiplication by τ ; here $\phi^* \mathcal{M}$ denotes \mathcal{M} with the Λ_Y -module structure on \mathcal{M} given by $\rho_{\phi^* \mathcal{M}} = \rho_{\mathcal{M}} \circ \phi_*$, where $\phi_* : \Lambda_Y \rightarrow \Lambda_Y$ is the automorphism induced by the homeomorphism $\phi : Y \rightarrow Y$. Thus, identifying the morphism (6-26), we obtain the following result:

6.12.1. Theorem. *Let $K = K_\phi$ be the mapping torus of a homeomorphism $\phi : Y \rightarrow Y$ as above and let \mathcal{M} be a $\Lambda = \mathbb{C}[\pi]$ -module in the extended category $\mathcal{E}(\mathcal{C}_A)$, where $\pi = \pi(K)$. Then there exists the following exact sequence*

$$\begin{aligned} \dots \rightarrow \mathcal{H}^{i-1}(Y, \mathcal{M}) \rightarrow \mathcal{H}^i(K, \mathcal{M}) \xrightarrow{i^*} \\ \xrightarrow{i^*} \mathcal{H}^i(Y, \mathcal{M}) \xrightarrow{\text{id} - \tau^{-1} \phi^*} \mathcal{H}^i(Y, \mathcal{M}) \rightarrow \mathcal{H}^{i+1}(K, \mathcal{M}) \rightarrow \dots \end{aligned} \quad (6-28)$$

of objects and morphisms of the extended category $\mathcal{E}(\mathcal{C}_A)$. In this exact sequence, $\phi^* : \mathcal{H}^i(Y, \mathcal{M}) \rightarrow \mathcal{H}^i(Y, \phi^* \mathcal{M})$ denotes the induced morphism (6-14), and $\tau : \mathcal{H}^i(Y, \mathcal{M}) \rightarrow \mathcal{H}^i(Y, \phi^* \mathcal{M})$ denotes "the coefficient" morphism induced by isomorphism (6-27). \square

6.12.2. Consider now a special case of the previous Theorem assuming that *the fundamental group $\pi(Y)$ acts trivially on \mathcal{M} .*

Then $\mathcal{H}^i(Y, \mathcal{M})$ equals to $H^i(Y, \mathbb{C}) \otimes_{\mathbb{C}} \mathcal{M}$. We also observe that the module $\phi^* \mathcal{M}$ is now identical with \mathcal{M} and the induced map $\phi^* : \mathcal{H}^i(Y, \mathcal{M}) \rightarrow \mathcal{H}^i(Y, \mathcal{M})$ can be identified with the tensor product of the induced map on the ordinary cohomology times the identity on \mathcal{M} .

We have given a left $\mathbb{C}[\tau, \tau^{-1}]$ -module structure on some $\mathcal{M} \in \text{ob}(\mathcal{E}(\mathcal{C}_A))$. Also, we may view the ordinary homology and cohomology of Y as left modules over the ring $\mathbb{C}[\tau, \tau^{-1}]$ with τ acting by means of the homomorphism ϕ_* induced by homeomorphism ϕ . Having this in mind, we will denote the kernel and cokernel of the following morphism of $\mathcal{E}(\mathcal{C}_A)$

$$1 \otimes \tau - \phi^* \otimes 1 : H^i(Y) \otimes_{\mathbb{C}} \mathcal{M} \rightarrow H^i(Y) \otimes_{\mathbb{C}} \mathcal{M} \quad (6-29)$$

by

$$\text{Hom}_{\mathbb{C}[\tau, \tau^{-1}]}(H_i(Y), \mathcal{M}) \quad \text{and} \quad \text{Ext}_{\mathbb{C}[\tau, \tau^{-1}]}(H_i(Y), \mathcal{M}) \quad (6-30)$$

correspondingly. Note that these notations can be justified.

Now, we may rewrite the exact sequence (6-28) in the following form

$$0 \rightarrow \text{Ext}_{\mathbb{C}[\tau, \tau^{-1}]}(H_{i-1}(Y), \mathcal{M}) \rightarrow \mathcal{H}^i(K, \mathcal{M}) \rightarrow \text{Hom}_{\mathbb{C}[\tau, \tau^{-1}]}(H_i(Y), \mathcal{M}) \rightarrow 0. \quad (6-31)$$

6.12.3. In the situation of the previous subsection assume that $\mathcal{M} \in \text{ob}(\mathcal{C}_A)$ (i.e. \mathcal{M} is projective) and that the morphism $\tau : \mathcal{M} \rightarrow \mathcal{M}$ has no discrete spectrum.

Then, as it is easy to see, $\text{Hom}_{\mathbb{C}[\tau, \tau^{-1}]}(H_i(Y), \mathcal{M})$ always vanishes and we obtain the following result:

6.12.4. Corollary. *Under the assumptions of Theorem 6.12.1, suppose additionally that $\mathcal{M} \in \text{ob}(\mathcal{C}_A)$ is such that $\pi(Y)$ acts on it trivially and the isomorphism $\tau : \mathcal{M} \rightarrow \mathcal{M}$ has no discrete spectrum. Then the extended cohomology of the mapping torus $K = K_\phi$ is given by the following object of the extended category $\mathcal{E}(\mathcal{C}_A)$*

$$\mathcal{H}^i(K, \mathcal{M}) = (1 \otimes \tau - \phi^* \times 1 : H^{i-1}(Y) \otimes_{\mathbb{C}} \mathcal{M} \rightarrow H^{i-1}(Y) \otimes_{\mathbb{C}} \mathcal{M}). \quad (6-32)$$

6.12.5. Example. Let Z be a compact Hausdorff space and let μ be a positive Radon measure on Z such that the measure of any non-empty open subset U of Z is positive. Consider the von Neumann category \mathcal{C}_A of representations of the algebra $L^\infty_{\mathbb{C}}(Z, \mu)$ of essentially bounded measurable functions on Z , which was discussed in section 2.6 (example 7).

The simplest example of a $\mathbb{C}[\tau, \tau^{-1}]$ -module in this category is as follows. Let \mathcal{M} be $L^2(Z, \mu)$ and let $\tau : \mathcal{M} \rightarrow \mathcal{M}$ be the operator of multiplication by a function $\tau : Z \rightarrow \mathbb{C}$, where $\tau, \tau^{-1} \in L^\infty_{\mathbb{C}}(Z, \mu)$.

Then the condition that τ has no discrete spectrum is equivalent to the assumption that *all level sets $\{\xi \in \mathbb{Z}; \tau(\xi) = c\}$ have zero measure.*

Formula (6-32) gives an explicit representation of the extended cohomology. The answer depends only on the structure of the Jordan decomposition of the induced morphism $\phi_* : H_{i-1}(Y) \rightarrow H_{i-1}(Y)$ on the ordinary homology with complex coefficients.

In particular, using Theorem 4.10 we obtain that *if the function τ is continuous and all the level sets have measure zero, then the i -dimensional extended cohomology $\mathcal{H}^i(K, \mathcal{M})$ of the mapping torus K is nonzero if and only if the spectrum of the induced map $\phi_* : H_{i-1}(Y) \rightarrow H_{i-1}(Y)$ has nonempty intersection with the set of values of function $\tau, \tau(Z) \subset \mathbb{C}$.*

As a conclusion we obtain:

if the induced map $\phi_ : H_{i-1}(Y) \rightarrow H_{i-1}(Y)$ is semi-simple, then studying extended cohomology of the mapping torus, one may recover all information about the usual homology of Y with complex coefficients and the action of the homeomorphism ϕ_* on it.*

This should be compared with the fact that the reduced L^2 cohomology of the mapping torus vanishes in many cases, cf. [L1].

Note also that using Theorem 6.12.1 inductively one may hopefully calculate the extended cohomology of abelian groups and of nilpotent groups.

§7. De Rham theorem for extended cohomology

The purpose of this section is to define De Rham version of the extended L^2 cohomology and to prove a De Rham type theorem, establishing isomorphism between combinatorially and analytically defined objects.

It should be noticed that some important consequences of this theorem are already well known: they are theorem of J.Dodziuk [D] (which gives equality of the von Neumann dimensions) and theorem of A. Efremov [E], [E1] (which establishes equivalence of the spectral density functions and equality of the Novikov-Shubin invariants). Since we know that the isomorphism type of a torsion object is not determined by the spectral density function, there is still a need to compare fully the combinatorially and analytically defined objects of the extended abelian category.

To my mind, the main part of [E], [E1] are written without control on the category, where the arguments are performed. This creates difficulties while trying to adopt the result of [E], [E1] in the framework of the present approach. As an example, let me mention that admitting densely defined operators, as morphisms (as in [E], [E1]) immediately makes all injective with dense image operators invertible and so completely eliminates the torsion phenomenon. I do not exclude possibility that the arguments of [E], [E1] could be rewritten and presented in a different form to avoid the categorical difficulties. Instead of doing this, I suggest in this section arguments (using sheaves and spectral sequences) which are much easier and more transparent, than the original arguments of [D] and [E], [E1].

I should mention that a new preprint of M. Shubin [S] contains a different version of De Rham type theorem for extended L^2 cohomology.

A few words about notations. In this section \mathcal{A} will denote a fixed algebra with involution and we will denote by $\mathcal{C}_{\mathcal{A}}$ the von Neumann category of all Hilbertian representations of \mathcal{A} , cf. §5. We will need also a category containing $\mathcal{C}_{\mathcal{A}}$, which we will denote by $\mathcal{F}_{\mathcal{A}}$. Its objects are all Fréchet topological vector spaces supplied with an action of the algebra \mathcal{A} (no condition involving the involution) and the morphisms are continuous \mathcal{A} -linear maps.

7.1. The De Rham complex. First, we are going to discuss the notion of *flat bundle* with fiber a Hilbertian module. This notion is quite similar to the standard notion of flat finite dimensional vector bundle, or to the notion of a flat Hilbert bundle over a von Neumann algebra, which was considered in [BFKM], for example; cf. also [CFM]. Roughly, flat bundle is a bundle with discrete structure group.

Let \mathcal{M} denote a fixed Hilbertian representation of \mathcal{A} . This module \mathcal{M} will serve as the typical fiber of our flat bundles. $GL(\mathcal{M})$ will denote the group of all $\mathcal{C}_{\mathcal{A}}$ -automorphisms of \mathcal{M} , i.e. all linear homeomorphisms $\mathcal{M} \rightarrow \mathcal{M}$ commuting with the \mathcal{A} -action.

Let X be a topological space. A *flat Hilbertian bundle over X with fiber \mathcal{M}* can be defined by a 1-cocycle on an open covering \mathcal{U} of X with values in the group $GL(\mathcal{M})$, i.e. by a function which associates an element $g_{UV} \in GL(\mathcal{M})$ for any ordered pair of open sets $U, V \in \mathcal{U}$ having nonempty intersection. It satisfies the following conditions: (a) $g_{UU} = 1$ and (b) $g_{UV} \cdot g_{VW} = g_{UW}$ if the intersection of the sets U, V and W is nonempty. As usual, given a 1-cocycle as above, one can construct the space of the bundle \mathcal{E} identifying in the disjoint union of the spaces $U \times \mathcal{M}$ the points $(x, m) \in U \times \mathcal{M}$ with $(x, g_{UV} \cdot m) \in V \times \mathcal{M}$ for x belonging to the intersection of U and V . The space \mathcal{E} admits a natural projection onto X . The flat Hilbertian bundle as above will be denoted by one letter \mathcal{E} . There is a well-known equivalence relation between the 1-cocycles.

If $X' \subset X$ is a subset, there is naturally defined the restriction of a flat Hilbertian bundle \mathcal{E} over X , onto X' , which is denoted $\mathcal{E}|_{X'}$ and is a flat Hilbertian bundle over X' .

Suppose now that X is a finite-dimensional smooth manifold (not necessarily compact). Denote by $C_{\mathcal{M}}^{\infty}(X)$ the Fréchet space of the smooth functions on X with values in \mathcal{M} . The family of Fréchet spaces $C_{\mathcal{M}}^{\infty}(U)$, where U runs over all open subsets of X together with the natural restriction maps, form a sheaf with values in the category $\mathcal{F}_{\mathcal{A}}$, which will be denoted by $C_{\mathcal{M}}^{\infty}$.

We want to study *smooth differential forms on X with values in the flat Hilbertian bundle \mathcal{E}* over X , cf. above. If \mathcal{E} is represented by a 1-cocycle with respect to a finite open covering $\mathcal{U} = \{U\}$ of X , then for each open set $U \in \mathcal{U}$ define $A^i(U, \mathcal{E})$ as

$$A^i(U, \mathcal{E}) = \Lambda^i(U) \otimes C_{\mathcal{M}}^\infty(U), \quad (7-1)$$

where $\Lambda^i(U)$ denotes the space of smooth i -forms on U with complex values, and the tensor product is taken over the ring $C^\infty(U)$. A *differential i -form on X with values in \mathcal{E}* is defined as a collection

$$\omega = = \{\omega_U\}_{U \in \mathcal{U}}, \quad (7-2)$$

where $\omega_U \in A^i(U, \mathcal{E})$ such that for any pair $U, V \in \mathcal{U}$ with nonzero intersection we have

$$\omega_U|_{U \cap V} = g_{UV} \cdot \omega_V|_{U \cap V}. \quad (7-3)$$

The space of all smooth i forms on X with values in \mathcal{E} will be denoted $A^i(X, \mathcal{E})$. This space has a natural Fréchet topology. Also, the initial algebra \mathcal{A} acts on $A^i(X, \mathcal{E})$.

There is *the operator of covariant derivative*

$$\nabla : A^i(X, \mathcal{E}) \rightarrow A^{i+1}(X, \mathcal{E}), \quad (7-4)$$

defined as follows. If $\omega \in A^i(X, \mathcal{E})$ is given by $\omega = = \{\omega_U\}_{U \in \mathcal{U}}$, then $\nabla(\omega) = \{d\omega_U\}_{U \in \mathcal{U}}$, where d denotes the usual exterior derivative. The operator ∇ is clearly well defined, continuous, commutes with the \mathcal{A} -action and satisfies $\nabla^2 = 0$.

We obtain the cochain complex in $\mathcal{F}_{\mathcal{A}}$, the *De Rham complex with values in \mathcal{E}* :

$$0 \rightarrow A^0(X, \mathcal{E}) \xrightarrow{\nabla} A^1(X, \mathcal{E}) \xrightarrow{\nabla} \dots A^i(X, \mathcal{E}) \xrightarrow{\nabla} A^{i+1}(X, \mathcal{E}) \xrightarrow{\nabla} \dots A^n(X, \mathcal{E}) \rightarrow 0, \quad (7-5)$$

where n is the dimension of the manifold X . We will denote the De Rham complex by $(A^*(X, \mathcal{E}), \nabla)$ for short.

7.2. Extended L^2 Čech cohomology. Suppose that \mathcal{E} is a flat Hilbertian bundle over a closed manifold X . Let \mathcal{U} be a *good* finite open covering of X (cf. [BT], page 42) and suppose that the flat bundle \mathcal{E} is trivial over every open set $U \in \mathcal{U}$. We will view now the flat bundle \mathcal{E} as a sheaf of its flat sections; it is defined as follows

$$\Gamma_{\mathcal{E}}(W) = \ker[\nabla : A^0(W, \mathcal{E}) \rightarrow A^1(W, \mathcal{E})] \quad \text{for any open subset } W \subset X. \quad (7-6)$$

Note that $\Gamma_{\mathcal{E}}(W)$ is isomorphic to \mathcal{M} if $\mathcal{E}|_W$ is trivial. We obtain in this way the space of Čech cochains $C^p(\mathcal{U}; \mathcal{E})$ (defined in the standard way) and the Čech cochain complex

$$\dots \xrightarrow{\delta} C^p(\mathcal{U}; \mathcal{E}) \xrightarrow{\delta} C^{p+1}(\mathcal{U}, \mathcal{E}) \xrightarrow{\delta} \dots \quad (7-7)$$

which we will denote by $(C^*(\mathcal{U}; \mathcal{E}), \delta)$. Observe that the spaces of cochains $C^p(\mathcal{U}; \mathcal{E})$ are naturally defined as *Hilbertian modules and the Čech boundary homomorphisms are continuous linear maps*. Thus, the Čech complex can be viewed as a complex in category $\mathcal{C}_{\mathcal{A}}$. Moreover, if the fiber \mathcal{M} is finite, i.e. $\mathcal{M} \in \text{ob}(\mathcal{C}_{\mathcal{A}}^{fin})$, then the whole Čech complex $C^p(\mathcal{U}; \mathcal{E})$ belongs to $\mathcal{C}_{\mathcal{A}}^{fin}$.

In particular, it follows that we may define *the Čech version of the extended L^2 cohomology* (which we will denote by $\check{\mathcal{H}}^*(\mathcal{U}, \mathcal{E})$) as the cohomology (understood in $\mathcal{E}(\mathcal{C}_A)$ or in $\mathcal{E}(\mathcal{C}_A^{fin})$) of the above Čech complex. It follows from Theorem 7.3 below that the Čech extended cohomology $\check{\mathcal{H}}^*(\mathcal{U}, \mathcal{E})$ does not depend on the (finite good open) cover \mathcal{U} .

Note that the construction of the extended L^2 cohomology used in §6 (based upon cell decompositions) is a special case of this Čech construction. Namely, choose a sufficiently fine C^1 tringulation of the manifold X and consider the covering \mathcal{U} of X formed by the stars of the vertices. Then it is easy to see that the Čech cohomology $\check{\mathcal{H}}^*(\mathcal{U}, \mathcal{E})$ of this covering will coincide with $\mathcal{H}^*(X, \mathcal{M})$, defined as in §6, using the monodromy representation of the flat bundle \mathcal{E} .

7.3. Theorem. (Čech - De Rham Theorem). *Suppose that X is a closed smooth manifold and \mathcal{E} is a flat Hilbertian bundle over X with fiber a Hilbertian module \mathcal{M} over \mathcal{A} ; let $(A^*(X, \mathcal{E}), \nabla)$ be the corresponding De Rham complex. Suppose that \mathcal{U} is a finite good (cf. [BT]) open cover of X and let $C^*(\mathcal{U}, \mathcal{E})$ be the corresponding Čech complex, viewed as a complex in category \mathcal{C}_A of Hilbertian modules. Then there is a chain homotopy equivalence*

$$(C^*(\mathcal{U}, \mathcal{E}), \delta) \rightarrow (A^*(X, \mathcal{E}), \nabla) \quad (7-8)$$

in \mathcal{F}_A .

In particular, we obtain, that the Čech extended L^2 cohomology $\mathcal{H}^*(\mathcal{U}, \mathcal{E})$ does not depend on the choice of a covering \mathcal{U} . We will mention some other corollaries later, cf. 7.10, 7.11.

The proof of Theorem 7.3 (which is given in 7.7) will be quite similar to the proof of the standard De Rham theorem given in [BT] for example. The main difference will be in the fact that we are not allowed to use spectral sequences in category \mathcal{F}_A (since it is only additive and not abelian). Instead, we will use Lemma 7.6.

The main step is the Poincaré lemma.

7.4. Lemma. (Poincaré lemma). *Let X be diffeomorphic to \mathbb{R}^n . Let M be a finitely generated Hilbertian module and let \mathcal{E} denote the trivial flat Hilbertian bundle with fiber M over X . Then there is a homotopy equivalence*

$$F : (\mathcal{M}, 0) \rightarrow (A^*(X, \mathcal{E}), \nabla) \quad (7-9)$$

between cochain complexes in category \mathcal{F}_A . Here $(\mathcal{M}, 0)$ denotes the cochain complex, which has only one nonzero chain space \mathcal{M} in dimension zero. The map $F : \mathcal{M} \rightarrow A^0(X)$ is given by assigning to a point $m \in \mathcal{M}$ the constant section with value m .

Proof. The proof of the Poincaré Lemma repeats the arguments of [BT], pages 34, 35 with an additional attention to the fact that the constructed in [BT] standard chain homotopies are actually morphisms of category \mathcal{F}_A . \square

7.5. Double complexes in \mathcal{F}_A . For the proof of Theorem 7.3 we will need some properties of double complexes in category \mathcal{F}_A . They have in fact a very general nature and are valued in arbitrary additive categories. In abelian categories they trivially follow from existence of two spectral sequences of a double complex.

Consider a *double complex* $K = \oplus_{p,q} K^{pq}$ in category \mathcal{F}_A having two differentials D' , of bidegree $(1, 0)$, and D'' , of bidegree $(0, 1)$, which satisfy $D'^2 = 0$, $D''^2 = 0$ and $D'D'' + D''D' = 0$. The *total differential* $D = D' + D''$ then satisfies $D^2 = 0$. We will always assume that all our double complexes are *positive and bounded* which means that there exists some integer N such that K^{pq} vanishes if one of p and q is negative or larger than N . Because of this condition we may consider *the total complex* $\text{Tot}(K) = K^n$, where $K^n = \oplus_{p+q=n} K^{pq}$, considered with the total differential D .

If K and L are two such double complexes, a *morphism* $\phi : K \rightarrow L$ is a collection of morphisms $K^{pq} \rightarrow L^{pq}$ in \mathcal{F}_A , which commute with the differentials D' and D'' . Morphism of double complexes clearly determines a chain map of the total complexes. We are interested in conditions on ϕ which guarantee that this chain map of the total complexes is a chain homotopy equivalence.

7.6. Lemma. *Let K and L be positive and bounded double complexes in \mathcal{F}_A . Suppose that $\phi : K \rightarrow L$ is a morphism between these double complexes satisfying the following condition: for each integer p the induced chain map of the vertical complexes $(K^{p,\bullet}, D'') \rightarrow (L^{p,\bullet}, D'')$ is a homotopy equivalence. Here $(K^{p,\bullet} = \oplus_q K^{pq}$ and similarly for L). Then the induced by ϕ morphism of the total chain complexes $(\text{Tot}(K), D) \rightarrow (\text{Tot}(L), D)$ is a chain homotopy equivalence.*

The same conclusion is true if we assume that for every q the induced by ϕ chain map of the horizontal complexes $(K^{\bullet,q}, D') \rightarrow (L^{\bullet,q}, D')$ is a homotopy equivalence.

Proof. Suppose first that our double complexes have only two nontrivial columns, i.e. $K^{pq} = 0$ and $L^{pq} = 0$ for all $p \neq i, i+1$. Then the double complex K consists of two cochain complexes $K^{i,\bullet}$ and $K^{i+1,\bullet}$ (two columns) and the horizontal differential $D' : K^{i,\bullet} \rightarrow K^{i+1,\bullet}$ can be viewed as a chain map between them. Then the total complex coincides with the cone of this chain map; similarly for L . In this form Lemma 7.6 is well known: it is one of the basic properties of triangulated categories; cf. for example [KS], Corollary 1.5.5.

The general case can be reduced to the case of two columns using induction. Namely, suppose $K^{i,\bullet}$ and $L^{i,\bullet}$ are the last nontrivial columns, i.e. $K^{pq} = 0$ and $L^{pq} = 0$ for $p > i$. Let K' be the total complex of the truncated double complex $\oplus_{p < i} K^{pq}$; similarly we will construct L' . Then by induction we know that the induced by ϕ chain map $K' \rightarrow L'$ is a chain homotopy equivalence. The horizontal differential D' gives us the maps $D' : K' \rightarrow K^{i,\bullet}$ and $D' : L' \rightarrow L^{i,\bullet}$ and the cones of these maps coincide with the total complexes $\text{Tot}(K)$ and $\text{Tot}(L)$ correspondingly. Thus we may again apply the Lemma in the case of two columns and the result follows. \square

7.7. Proof of theorem 7.3. For any open set $U \subset X$ we may consider the space of smooth q -forms on U with values in \mathcal{E} , which we denote now $A_{\mathcal{E}}^q(U)$; this defines a sheaf $A_{\mathcal{E}}^q$ over X with values in category \mathcal{F}_A . The operator of covariant derivative defines a homomorphism of sheaves $\nabla : A_{\mathcal{E}}^q \rightarrow A_{\mathcal{E}}^{q+1}$.

Fix a good finite open cover \mathcal{U} of X .

Consider the following double complex $K = \oplus_{pq} K^{pq}$, where $K^{pq} = C^p(\mathcal{U}, A_{\mathcal{E}}^q)$ is the space of Čech cochains of cover \mathcal{U} with values in sheaf $A_{\mathcal{E}}^q$; this construction was called in [BT] the Čech - De Rham complex. We have two differentials in this double complex: $D' = \delta$ is the Čech differential, and $D'' = (-1)^p \nabla$ on K^{pq} is the covariant

derivative. These two differentials satisfy the necessary commutation relations. Note that the constructed double complex is positive and bounded.

We will construct now two more double complexes. The first one, which we will denote by A , is obtained from the De Rham complex. Namely, define A^{pq} to be 0 if $p \neq 0$ and $A^{0q} = A_{\mathcal{E}}^q(X)$. The horizontal differential D' will be zero and the vertical D'' coincides with ∇ .

The third double complex will be denoted by \check{C} , it is essentially the Čech complex. It is defined by $\check{C}^{pq} = 0$ for $q \neq 0$ and $\check{C}^{p0} = C^p(\mathcal{U}, \mathcal{E})$. Note that \mathcal{E} is considered here as the sheaf of its flat sections. The horizontal differential of \check{C} is the Čech boundary δ and the vertical differential is zero.

There are two natural morphisms of the constructed double complexes

$$\phi : A \rightarrow K \quad \text{and} \quad \psi : \check{C} \rightarrow K$$

given as follows: $\phi^{0q} : A^{0q} \rightarrow C^0(\mathcal{U}, A_{\mathcal{E}}^q)$ acts by $\phi^{0q}(\omega) = ga = \{\omega|_U\}_{U \in \mathcal{U}}$ and the morphisms on all other places are zero. The second morphism ψ is the natural inclusion $\psi^{p0} : C^{p0} = C^p(\mathcal{U}; \mathcal{E}) \rightarrow C^p(\mathcal{U}; A_{\mathcal{E}}^0)$ and zero on all other places.

Our theorem 7.3 will follow from Lemma 7.6 (applied twice) if we will establish two the following statements:

- (1) for each integer q the induced horizontal chain map $\phi^{\bullet q} : A^{\bullet q} \rightarrow K^{\bullet q}$ is a chain homotopy equivalence in $\mathcal{F}_{\mathcal{A}}$;
- (2) for each integer p the induced vertical chain map $\psi^{p\bullet} : \check{C}^{p\bullet} \rightarrow K^{p\bullet}$ is a chain homotopy equivalence in $\mathcal{F}_{\mathcal{A}}$.

However, (2) is guaranteed by the Poincaré lemma 7.4 and (1) is given by the chain homotopy constructed in [BT], page 94 using a partition of unity. This completes the proof. \square

7.8. De Rham version of the extended cohomology. Unfortunately, one cannot use the De Rham complex (7-5) in order to define the De Rham version of the extended cohomology. The reason is that (7-5) is a complex in category $\mathcal{F}_{\mathcal{A}}$ of Fréchet representations of \mathcal{A} , and the abelian extension constructions of §1 and §5 do not apply to it. In order to overcome this difficulty, we consider here another complex (the Sobolev - De Rham complex) which is homotopy equivalent to (7-5) and belongs to category $\mathcal{C}_{\mathcal{A}}$ of all Hilbertian representations of \mathcal{A} ; note that $\mathcal{C}_{\mathcal{A}}$ admits an extended abelian category $\mathcal{E}(\mathcal{C}_{\mathcal{A}})$, which was constructed in §5.

If X is a closed smooth manifold and \mathcal{E} is a flat Hilbertian bundle over X , we will denote by $A_{(m)}^p(X, \mathcal{E})$ the m -th Sobolev space of p -forms on X with values in \mathcal{E} (here m is a non-negative integer). This space is defined as the completion of the space of smooth forms $A^p(X, \mathcal{E})$ (which appeared earlier) with respect to the m -th Sobolev's norm $\|\cdot\|_{(m)}$. The latter is defined in the usual way by using partitions of unity.

The covariant derivative ∇ defines a continuous operator $\nabla : A_{(m)}^p(X, \mathcal{E}) \rightarrow A_{(m-1)}^{p+1}(X, \mathcal{E})$. Thus we obtain *the m -th Sobolev - De Rham complex*

$$0 \rightarrow A_{(m)}^0(X, \mathcal{E}) \cdots \rightarrow A_{(m+1-p)}^{p-1}(X, \mathcal{E}) \xrightarrow{\nabla} A_{(m-p)}^p(X, \mathcal{E}) \xrightarrow{\nabla} A_{(m-1-p)}^{p+1}(X, \mathcal{E}) \rightarrow \dots \quad (7-10)$$

Here $m \geq n = \dim X$.

The complex (7-10) was studied earlier in [D] and in [LL] in special cases.

Note that (7-10) is a chain complex in category $\mathcal{C}_{\mathcal{A}}$, i.e. it consists of Hilbertian spaces (cf. [P]) and continuous linear maps commuting with \mathcal{A} . Thus, using the results of §5, we may define *the De Rham version of the extended L^2 cohomology*

$$\mathcal{H}_{(m-p)}^p(K, \mathcal{E}) = (\nabla : A_{(m+1-p)}^{p-1}(X, \mathcal{E}) \rightarrow Z_{(m-p)}^p), \quad (7-11)$$

where $Z_{(m-p)}^p = \ker[\nabla : A_{(m-p)}^p(X, \mathcal{E}) \rightarrow A_{(m-p-1)}^{p+1}(X, \mathcal{E})]$, i.e. the cohomology of (7-10) in the extended category $\mathcal{E}(\mathcal{C}_{\mathcal{A}})$.

The following theorem implies, in particular, that the extended cohomology of the Sobolev - De Rham complex (7-10) does not depend on m .

7.9. Theorem. (Sobolev - De Rham Theorem). *Let X be a closed smooth manifold with a flat Hilbertian bundle $\mathcal{E} \rightarrow X$. Then the m -th Sobolev - De Rham complex (7-10) is homotopy equivalent in category $\mathcal{C}_{\mathcal{A}}$ to the Čech complex $(C^p(\mathcal{U}; \mathcal{E}), \delta)$ constructed using any finite good open cover \mathcal{U} of X .*

Proof. We will show that the natural inclusions $i : A^p(X, \mathcal{E}) \rightarrow A_{(m-p)}^p(X, \mathcal{E})$ determine a chain homotopy equivalence i in $\mathcal{F}_{\mathcal{A}}$ between the De Rham complex (7-5) and the m -th Sobolev - De Rham complex (7-10); this together with Theorem 7.3 will complete the proof of Theorem 7.9.

Fix a Riemannian metric on X and a Hermitian metric on \mathcal{E} . Let $\Delta_p = \nabla^* \nabla + \nabla \nabla^*$ denote the Laplacian acting on the spaces of p -forms. It determines a morphism $\Delta_p : A_{(m-p)}^p(X, \mathcal{E}) \rightarrow A_{(m-p-2)}^p(X, \mathcal{E})$ in $\mathcal{C}_{\mathcal{A}}$ for any m . Consider the heat operator $e^{-\Delta_p}$. It is an infinitely smoothing operator, it maps any Sobolev space $A_{(m-p)}^p(X, \mathcal{E})$ into the space of smooth forms $A^p(X, \mathcal{E})$, cf. for example, [BFKM].

We denote by $j : A_{(m-p)}^p(X, \mathcal{E}) \rightarrow A^p(X, \mathcal{E})$ the map given by $j(\omega) = e^{-\Delta_p}(\omega)$. It commutes with the covariant derivative ∇ and determines a chain map in category $\mathcal{F}_{\mathcal{A}}$ between the Sobolev and smooth De Rham complexes.

We want to show that i and j are mutually inverse homotopy equivalences. To do so it is enough to construct an operator H which maps continuously any Sobolev space $A_{(m')}^p(X, \mathcal{E})$ into $A_{(m'+1)}^{p-1}(X, \mathcal{E})$ and satisfies the homotopy relation

$$j(\omega) - \omega = \nabla H + H \nabla, \quad \text{for any } \omega \in A_{(m')}^p(X, \mathcal{E}). \quad (7-12)$$

We define H as follows

$$H = H' \circ \nabla^*, \quad \text{where } H' = \frac{1}{2\pi i} \int_C \frac{\phi(\lambda)}{\lambda - \nabla^* \nabla} d\lambda, \quad (7-13)$$

and

$$\phi(\lambda) = \frac{e^{-\lambda} - 1}{\lambda}. \quad (7-14)$$

Here we view $\nabla^* \nabla$ as a self adjoint operator acting from the closure of the subspace of coclosed forms $\overline{\text{im}(\nabla^*)} \subset A_{(m-p)}^p(X, \mathcal{E})$ into itself, and being zero on the space orthogonal to $\overline{\text{im}(\nabla^*)}$. It's spectrum is real and non-negative and so the operator calculus applies. The integration curve C is $x + 1 = |y|$ on the plane (x, y) with the orientation chosen such that y decreases. The function $\phi(\lambda)$ decays as x^{-1} along the

curve C and the resolvent $(\lambda - \nabla^* \nabla)^{-1}$ also decays as x^{-1} ; this shows convergence of H in (7-13) as an operator from $A_{(m')}^p(X, \mathcal{E})$ to $A_{(m'+2)}^p(X, \mathcal{E})$.

The homotopy relation (7-12) can be easily verified. \square

We refer to the recent preprint [S] of M. Shubin which contain a different version of De Rham type theorem for extended cohomology.

7.10. Corollary. (J.Dodziuk [D], A.Efremov [E]) *Let \mathcal{C}_A^{fin} denote a full finite von Neumann subcategory of \mathcal{C}_A , cf. §5. Suppose that the fiber \mathcal{M} of the flat bundle \mathcal{E} is finite, i.e $\mathcal{M} \in \text{ob}(\mathcal{C}_A^{fin})$. Let tr be a trace on the category \mathcal{C}_A^{fin} , cf. §§2, 5. Then the von Neumann dimension with respect to the trace tr of the extended De Rham cohomology $\mathcal{H}_{(m-i)}^i(K, \mathcal{E})$ coincides with the von Neumann dimension of the extended Čech cohomology $\check{\mathcal{H}}^i(\mathcal{U}, \mathcal{E})$. Also, the spectral density functions of the torsion parts of $\mathcal{H}_{(m-i)}^i(K, \mathcal{E})$ and $\check{\mathcal{H}}^i(\mathcal{U}, \mathcal{E})$ are dilatationally equivalent and their Novikov-Shubin invariants coincide.*

7.11. Corollary. (Finiteness.) *As in 7.10, suppose that given a full finite von Neumann subcategory \mathcal{C}_A^{fin} of \mathcal{C}_A and that $\mathcal{M} \in \text{ob}(\mathcal{C}_A^{fin})$. Then the extended De Rham cohomology $\mathcal{H}_{(m-i)}^i(K, \mathcal{E})$ is also finite (i.e. it is isomorphic in $\mathcal{E}(\mathcal{C}_A)$ to an object of $\mathcal{E}(\mathcal{C}_A^{fin})$). This can also be stated by saying that the Sobolev - De Rham complex $(A_{(m-*)}^*(K, \mathcal{E}), \nabla)$ is Fredholm.*

This follows automatically from Theorem 7.9 since the Čech extended cohomology is clearly finite if \mathcal{M} is finite.

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